Existence and smoothness of continuous and discrete solutions of a two-dimensional shallow water problem over movable beds

B. Toumbou *, A. Mohammadian

Department of Civil Engineering, University of Ottawa, 161 Louis Pasteur, Ottawa, Ontario, K1N 6N5, Canada

ARTICLE INFO

Article history:
Received 30 August 2011
Accepted 30 August 2012
Communicated by S. Carl

Keywords:
Existence theorem
Shallow water equations
Movable beds
Smoothness
Galerkin method
Discrete system
Discontinuous Galerkin method

ABSTRACT

We consider a two-dimensional shallow water system over movable beds. We begin with a continuous system and prove the existence of the solutions, and then we investigate their smoothness. Then, we employ a Galerkin method to obtain a finite-dimensional problem which is solved using a Brouwer fixed point theorem. Therefore, we show that the limits of the resulting solution sequences satisfy the model equations.

After solving the continuous problem, we focus on the corresponding discrete problem. We employ a local discontinuous Galerkin scheme for numerical solution of the discrete system and conduct an error analysis of the numerical scheme. We prove that the method is convergent and that the error is bounded according to a specific norm defined herein.

© 2012 Elsevier Ltd. All rights reserved.

1. Introduction

Many natural systems include shallow flows. In such flows, the horizontal scales of the flow are much larger than the vertical ones, and the changes of physical properties of the flows in the vertical direction are much smaller than in the horizontal one. In particular, shallow flows are observed in most rivers, lakes, seas, and oceans. In most natural cases, the bed consists of movable materials and the topography changes in time due to erosion or sedimentation which leads to morphological changes. Thus, solution of shallow water over movable bed equations (SWMBEs) is an important issue in practical applications.

The mathematical study of the existence, uniqueness, and smoothness of the solutions for shallow water equations in different conditions, started two decades ago, has been improved since. The existence of solutions for a homogeneous system of shallow water equations (SWEs) was initially proved by Orenga [1]. Chatelon and Orenga [2] extended that study to a non-homogeneous system of SWEs. Later, Chatelon and Orenga [3] studied the smoothness and uniqueness of the solutions. Those existence, uniqueness, and smoothness results were then extended by Munoz-Ruiz et al. to one-dimensional [4] and two-dimensional [5] bi-layer SWEs.

On the numerical side, many finite volume and finite elements schemes have been developed for shallow water equations over fixed beds during the past two decades. In particular, discontinuous Galerkin schemes have recently become very popular for fluid mechanics [6]. This motivates the study of the performance of these schemes for SWMBEs.

The goal of this paper is to study two-dimensional SWMBEs. We study and prove the existence and the smoothness of the solutions in suitable chosen spaces. The relationship between the sediment discharge and flow velocity could be linear or nonlinear, depending on the type of sediments and their physical characteristics such as density and mean diameter. This paper focuses on the linear case. A study of the nonlinear case is currently in progress by the authors, and the extension of the
present results to that case will be the subject of subsequent papers. Moreover, here we consider a continuous–discontinuous Galerkin scheme for numerical solution of the system, and perform an error analysis of the numerical scheme. This paper is the first of its kind, to the best of our knowledge, for flows over movable beds.

This paper is organized as follows. In Section 2, we review shallow water equations over movable beds. In Section 3, we present the weak formulation of the problem. Section 4, proves the existence and the smoothness of the solutions in suitable chosen spaces. In Section 5, we present a continuous–discontinuous Galerkin scheme for numerical solution of the system, and we examine the convergence of the method by error analysis. Some concluding remarks complete the study.

2. The two-dimensional shallow water equations over movable beds

The problem under study is the following two-dimensional SWMBE in a non-conservative form:

\[
\begin{align*}
\partial_t u - A \Delta u + (u \cdot \nabla) u + g \nabla (h + z) &= 0, \quad \text{in } \mathcal{Q}, \\
\partial_t h + \nabla \cdot (u h) &= 0, \quad \text{in } \mathcal{Q}, \\
\partial_t z + \sigma \nabla \cdot q_z &= 0, \quad \text{in } \mathcal{Q}, \\
u &= 0, \quad \text{on } \partial \mathcal{Q} \times (0, T), \\
u(t = 0) &= u_0, \quad \text{in } \mathcal{Q}, \\
h(t = 0) &= h_0, \quad \text{in } \mathcal{Q}, \\
z(t = 0) &= z_0, \quad \text{in } \mathcal{Q},
\end{align*}
\] (1)

The unknowns \(h(x, t), u(x, t), \) and \(z(x, t)\) are defined on \(\mathcal{Q} = \mathcal{Q} \times [0, T]\), and they represent the flow depth, flow velocity, and the bed elevation at the section at the location of coordinate \(x\) at time \(t\). Moreover, \(\sigma = (1 - \lambda_p)^{-1}\), where \(\lambda_p\) is the sediment porosity and \(A, g,\) and \(q_z(u)\) are the constant coefficient of viscosity, gravitational acceleration, and the sediment transport rate per unit width, respectively. The latter is a function of the flow velocity in the channel, which is written as

\[
q_z(u) = C_u u \|u\|^{m-1},
\] (2)

where \(C_u\) and \(m\) are empirical coefficients which depend on the sediment materials. Eq. (2) implies that there is a direct relationship between the flow velocity and the sediment transport rate, and the sediment is transported in the direction of water flow. Depending on sediment physical characteristics such as shape, mean diameter, and density, this relationship could be linear or nonlinear. In this paper, a linear case \((m = 1)\) is considered and studied, which is commonly used in practice (see, e.g., [7, Section 4.1]). A study of the nonlinear case \((m > 1)\) is currently in progress by the authors.

3. Weak formulation

To begin with, let us consider the weak formulation of problem (1). In this paper, the following notation will be used.

\(\langle ., . \rangle\): the scalar product of \(L^2(\mathcal{Q})\) or \((L^2(\mathcal{Q}))^2\).

\(\| . \|\): their corresponding norm.

\(\| . \|_{W^{m,p}(\mathcal{Q})}\): the usual norm in the space \(W^{m,p}(\mathcal{Q})\).

Let \(V\) and \(V\) be respectively the spaces \(H^1_0(\mathcal{Q})\) and \((H^1_0(\mathcal{Q}))^2\) equipped with the norms

\[
\|u\|^2_V = \|u\|^2 + \|\nabla u\|^2,
\] (3)

and

\[
\|u\|^2_{\mathcal{V}} = \|u\|^2 + \|\nabla u\|^2,
\] (4)

or with the equivalent norms (note that \(\mathcal{Q}\) is bounded)

\[
\|u\|^2_{\mathcal{V}} = \|\nabla u\|^2,
\] (5)

and

\[
\|u\|^2_{\mathcal{V}} = \|\nabla u\|^2.
\] (6)

Furthermore, since \(\mathcal{Q}\) is bounded, the bilinear form given by

\[
a(u, v) = \langle \nabla u, \nabla v \rangle
\] (7)

is elliptic. We also define

\[
V_u := L^\infty(0, T; (L^2(\mathcal{Q}))^2) \cap L^2(0, T; \mathcal{V}) \cap L^2(0, T; (C^0(\mathcal{Q}))^2),
\] (8)

\[
V_h := L^\infty(0, T; L^1(\mathcal{Q})),
\] (9)

\[
V_z := L^\infty(0, T; L^1(\mathcal{Q})) \cap L^2(\mathcal{Q}).
\] (10)

The variational form of problem (1) is written as follows.
3.1. A priori estimates

Let us introduce a basis for $V$ denoted by $\{v_1, \ldots, v_n, \ldots\}$, whose elements belong to $(H^3(\Omega))^2$. Let $V_n$ be the set of linear combinations of the $n$ first elements of the basis. We consider the following problem.

Find $(u_0, h_n, z_n)$ in

$$
\left\{ \begin{array}{l}
(\partial_t u, v) + A(\nabla u, \nabla v) + ((u \cdot \nabla)u, v) - g(h, \nabla \cdot v) - g(z, \nabla \cdot v) = 0, \quad \forall v \in V, \\
\partial_t z + \sigma \nabla \cdot q_e = 0, \\
u(t = 0) = u_0 \in V, \\
h(t = 0) = h_0 > 0 \in L^2(\Omega), \\
z(t = 0) = z_0 \in L^2(\Omega).
\end{array} \right. $$

(11)

where the data and the constants satisfy the following condition:

$$
\|u_{0,n}\|^2 + 2g \int_\Omega h_{0,n} \log h_{0,n} \, dx + D \|z_{0,n}\|^2 + \frac{2g}{\varepsilon} \text{meas}(\Omega) < \left(\frac{2A - \lambda}{C}\right)^2,
$$

(13)

where $\lambda$ is a strictly positive real number sufficiently small and $D = (\sigma C_i)^{-1}$. The positive real number $C$ depends on the Gagliardo–Nirenberg constants defined in what follows.

Here we give some a priori estimates of the approximate solutions.

**Lemma 3.1.** If $(u_n, h_n, z_n)$ is a classical solution of problem $(V_n)$, then the following relations hold:

$$
h_n > 0, $$

(14)

$$
\int_\Omega h_n(t) \, dx = \int_\Omega h_{0,n} \, dx.
$$

(15)

$$
\int_\Omega z_n(t) \, dx = \int_\Omega z_{0,n} \, dx.
$$

(16)

$$
\|u_n(t)\|^2 + \int_0^n B(s)\|u_n(s)\|^2 \, ds + 2g \int_\Omega h_n(t) \log h_n(t) \, dx + gD\|z_n(t)\|^2 \leq \|u_{0,n}\|^2 + 2g \int_\Omega h_{0,n} \log h_{0,n} \, dx + gD\|z_{0,n}\|^2.
$$

(17)

where $B(s) = 2A - C\|u_n(s)\|$.

**Proof.** Relations (14) and (15) are shown in [1,8]. Relation (16) is easily obtained by integrating $(V_n)_3$ in $Q$. In order to obtain (17), we take $v = u_n$ in $(V_n)_3$:

$$
\frac{1}{2} \frac{d}{dt} \|u_n\|^2 + A\|u_n\|^2 - g(h_n, \nabla \cdot u_n) - g(z_n, \nabla \cdot u_n) = -((u_n \cdot \nabla)u_n, u_n).
$$

(18)

It can be shown that (see [1,8])

$$
g(h_n, \nabla \cdot u_n) = \frac{d}{dt}(h_n \log h_n - h_n, 1).
$$

(19)
By using (15), relation (19) leads to
\[
- g \int_0^1 (h_n, \nabla \cdot u_n) = g \int_\Omega h_n \log h_n \, dx - g \int_\Omega h_{0,n} \log h_{0,n} \, dx. \tag{20}
\]

By using the sediment continuity equation in (\(\gamma_n\))_1,
\[
- g(z_n, \nabla \cdot u_n) = gD(z_n, \partial_t z_n)
\]
\[
= \frac{gD}{2} \frac{d}{dt} (z_n^2, 1). \tag{21}
\]

Let us estimate the term in the right-hand side of (18). Since \( u = (u, v) \), we have
\[
((u_n \cdot \nabla)u_n, u_n) = \frac{1}{3} \int_\Omega (\partial_t u_n^3 + \partial_y v_n^3) \, dx + \frac{1}{2} \int_\Omega (u_n \partial_x v_n^2 + v_n \partial_y u_n^2) \, dx. \tag{22}
\]

By integrating by parts the second term in the right-hand side of (22), and by using the fact that \( u_n = (u_n, v_n) \in V_n \) for the first term in the right-hand side of (22), we obtain
\[
|((u_n \cdot \nabla)u_n, u_n)| \leq \frac{1}{2} \left( \| u_n \|_{L^1(\Omega)} \| u_n \|_V + \frac{1}{2} \| u_n \|_{L^1(\Omega)}^2 \| v_n \|_V. \tag{23}\right.
\]

Let \( C_1 \) and \( C_2 \) be best constants of the following Gagliardo–Nirenberg inequalities:
\[
\| u_n \|^2_{L^1(\Omega)} \leq C_1 \| u_n \|_V \| u_n \|_V; \tag{24}\]
\[
\| v_n \|^2_{L^1(\Omega)} \leq C_1 \| v_n \|_V \| v_n \|_V. \tag{25}\]

By using (24) and (25), we obtain
\[
|((u_n \cdot \nabla)u_n, u_n)| \leq \frac{1}{2} \| u_n \|_V \| u_n \|_V \left( C_1 \| u_n \|_V + C_2 \| v_n \|_V \right)
\]
\[
\leq \frac{1}{4} \max(C_1, C_2) \left( \| u_n \|_V^2 + \| v_n \|_V^2 \right) \left( \| u_n \|_V + \| v_n \|_V \right)
\]
\[
\leq \frac{\sqrt{7}}{4} \max(C_1, C_2) \left( \| u_n \|_V^2 + \| v_n \|_V^2 \right) \left( \| u_n \|_V^2 + \| v_n \|_V^2 \right)^{\frac{1}{2}}
\]
\[
\leq \frac{C}{2} \| u_n \|_V^2 \| u_n \|_V^2. \tag{26}\]

where \( C = \frac{\sqrt{7}}{4} \max(C_1, C_2) \).

Substituting (20), (21) and (26) in (18), we obtain (17). \( \square \)

4. An existence theorem

In this section, a global existence result with controlled data is presented.

**Theorem 4.1.** Let \( u_0 \in V \) and \( (h_0, h_0 \log(h_0)) \in L^2(\Omega) \times L^1(\Omega), h_0 > 0 \) and \( z_0 \in L^2(\Omega) \) verify relation (13). Then problem (V) has a solution \( (u, h, z) \) that satisfies the following estimate:
\[
\| u \|^2_{L^\infty(0,T;L^2(\Omega))^2} + B \| u \|^2_{L^2(0,T;V)} + 2g \| h \log h \|^2_{L^\infty(0,T;L^1(\Omega))} + gD \| z \|^2_{L^\infty(0,T;L^2(\Omega))} \leq M, \tag{27}\]

where \( M \) depends solely on the initial data and \( B \) is defined from the time-dependent coefficient \( B_n(s) \) in (17) as
\[
B_n = \inf_{t \in [0,T]} B_n(t) \quad \text{and} \quad B = \inf_{n \in \mathbb{N}} B_n. \tag{28}\]

We first show that the constant \( B \) in (27) verifies
\[
B > \lambda > 0.
\]
Indeed, by using (13), we obtain \( \|u_{0,n}\| < \frac{2A - \lambda}{C} \). Since \( u_n \) is continuous in \([0, T]\), there exists \( t_1 > 0 \) such that
\[
\|u_n(t)\| < \frac{2A - \lambda}{C}, \quad \forall t \in [0, t_1] \text{ and } \forall n \in \mathbb{N}.
\]
Consequently, we have
\[
B_n(t) > \lambda > 0, \quad \forall t \in [0, t_1] \text{ and } \forall n \in \mathbb{N}.
\]
By setting \( T = t_1 \), it follows that
\[
\inf_{t \in [0, T]} B_n(t) > \lambda > 0, \quad \forall n \in \mathbb{N}.
\]
Finally, by using (28) and (29), we show that
\[
B > \lambda > 0.
\]

The proof of this theorem is split into several steps: we build a sequence of approximated solutions, Then we state and prove several lemmas that will allow us to pass to the limit into the continuity and momentum equations as in [5,8]. □

4.1. Approximate solutions

**Lemma 4.2.** Problem \( (\forall_n) \) has a solution \( \{u_n, h_n, z_n\} \) in
\[
\left[ L^\infty(0, T; (L^2(\Omega))^2) \cap L^2(0, T; V_n) \cap L^2(0, T; (e^0(\Omega))^2) \right] \times c^1(\overline{Q}) \times L^\infty(0, T; L^2(\Omega))
\]
which satisfies
\[
\|u_n\|^2_{L^\infty(0,T;L^2(\Omega)^2)} + \|u_n\|^2_{L^2(0,T;V)} + \|h_n\|_{L^\infty(0,T;L^1(\Omega))} + \|z_n\|^2_{L^\infty(0,T;L^2(\Omega))} < N,
\]
where \( N \) only depends on the initial data.

**Proof.** In order to prove this lemma, we apply the Brouwer fixed point theorem as in [8]. We obtain approximated solutions that satisfy the a priori estimates. In fact, using the relation (13) and following the same steps as in [8], and due to the regularity of the basis, we have \( u_n \in H^1(0, T; (H^3(l))^2) \). Therefore, \( u_n \in c^0([0, T]; (c^1(\Omega))^2) \). Finally, the estimations given in (32) are obtained as in Lemma 3.1.

Estimation (32) is obtained by combining (17) and (30), and by using the fact that
\[
h \log h \geq -\frac{1}{e}.
\]

4.2. Passage to the limit

In this section, we present a lemma that is used to pass to the limit in the approximated equations and to conclude the proof of the theorem. The passage to the limit is done by adapting the procedure developed in [2,8]. In that case, the most difficult point was to pass the limit in the continuity equation.

**Lemma 4.3.** For each \( n \in \mathbb{N} \), let
\[
\{u_n, h_n, z_n\} \in \left[ L^\infty(0, T; (L^2(\Omega))^2) \cap L^2(0, T; V_n) \cap L^2(0, T; (e^0(\Omega))^2) \right] \times L^\infty(0, T; L^2(\Omega)) \times L^\infty(0, T; L^2(\Omega))
\]
be the solution of \( (\forall_n) \) given by Lemma 4.2. Then we have
\[
u_n, h_n \text{ is bounded in } L^2(0, T; (L^1(\Omega))^2),
\]
\[
\partial_t u_n \text{ is bounded in } L^2(0, T; (H^{-3}(\Omega))^2),
\]
and we can extract from \( u_n \) and \( h_n \) subsequences still denoted \( u_n \) and \( h_n \) such that
\[
u_n \to u \text{ in } L^2(0, T; V) \text{ weakly,}
\]
\[
u_n \to u \text{ in } L^\infty(0, T; (L^2(\Omega))^2) \text{ weakly-star,}
\]
\[
h_n \to h \text{ in } L^\infty(0, T; L^1(\Omega)) \text{ weakly-star,}
\]
\[
h_n \log h_n \to h \log h \text{ in } L^\infty(0, T; L^1(\Omega)) \text{ weakly-star,}
\]
\[
(u_n \cdot \nabla) u_n \to (u \cdot \nabla) u \text{ in } L^\frac{4}{3}(Q) \text{ weakly,}
\]
\[
u_n, h_n \to uh \text{ in } L^1(Q) \text{ weakly,}
\]
\[
z_n \to z \text{ in } L^2(Q) \text{ weakly,}
\]
\[
z_n \to z \text{ in } L^\infty(0, T; L^2(\Omega)) \text{ weakly-star.}
\]
Proof. Results (36)–(38) are shown in [1]. The last two results, (42) and (43), are obtained directly using Lemma 3.1. It remains to prove (34)–(35) and (39)–(41) now. Let us prove (34).

\[
\int_\Omega |u_n h_n| \, dx \leq \|u_n\|_{L^\infty(\Omega)} \int_\Omega |h_n| \, dx.
\]  
(44)

Recall that \(u_n \in V_n\) and that the embedding from \(V_n\) in \((C^0(\Omega))^2\) is continuous. There exists \(M > 0\) independent of \(n\) such that

\[
\forall n \in \mathbb{N}, \quad \|u_n\|_{L^\infty(\Omega)} < M.
\]  
(45)

By using the continuity equation, we have

\[
\int_\Omega |h_n| \, dx = \int_\Omega |h_{0,n}| \, dx.
\]  
(46)

Then we deduce that \(u_n h_n\) is bounded in \(L^2(0, T; (L^1(\Omega))^2)\). We now prove (35). Indeed, for almost every \(t\), \(h_n(t) \in L^1(\Omega)\) which is continuously embedded in the dual of \(C^0(\Omega)\). Because the embedding from \(H^2(\Omega)\) to \(C^0\) is continuous in two dimensions, we deduce by duality that the imbedding from \(L^1(\Omega)\) to \(H^{-2}(\Omega)\) is also continuous, and hence \(h_n \in L^2(0, T; H^{-2}(\Omega))\). Consequently, this proves, by using the previous estimates, that

\[
\partial_t u_n \text{ is bounded in } L^\frac{4}{3}(0, T; (H^{-3}(\Omega))^2).
\]  
(47)

The proof of (39) is obtained by using (17) and the following basic inequality:

\[
h \log h \geq -\frac{1}{e} \quad \forall h \geq 0.
\]  
(48)

Let us prove (40). Let \(v\) be a regular function of \(D(Q)\).

\[
\left| \int_\Omega \left( \left( u_n(x, t) \cdot \nabla \right) u_n(x, t) - \left( u(x, t) \cdot \nabla \right) u(x, t) \right) \cdot v(x, t) \, dx \, dt \right|
\]
\[\leq \int_\Omega \left| \left( \left( u_n(x, t) - u(x, t) \right) \cdot \nabla \right) u_n(x, t) \right| \cdot v(x, t) \, dx \, dt + \int_\Omega \left| \left( u(x, t) \cdot \nabla \right) \left( u_n(x, t) - u(x, t) \right) \right| \cdot v(x, t) \, dx \, dt.
\]  
(49)

The first term in the right-hand side of (49) can be estimated as follows:

\[
\left| \int_\Omega \left( \left( u_n(x, t) - u(x, t) \right) \cdot \nabla \right) u_n(x, t) \right| \cdot v(x, t) \, dx \, dt \right| \leq \|\nabla u_n\| \|v\|_{L^\infty(\Omega)} \|u_n - u\|.
\]  
(50)

From (36), we deduce that the right-hand side of (50) goes to zero as \(n \to \infty\). The second term in the right-hand side of (49) goes to zero as \(n \to \infty\) due to (36). This ends the proof of (40).

The proof of (41) is completed by combining (34) and compactness results as in [1,8].

4.3. Proof of the theorem

Let \(u_0, h_0\), and \(z_0\) be the initial conditions of problem \(P\). Let \(\{u_{0,n}\}\) be a sequence with elements \(u_{0,n} \in V_n\) such that

\[
u_{0,n} \to u_0 \quad \text{in } V \text{ strongly}.
\]  
(51)

Also let \(\{h_{0,n}\}\) be a sequence in \(c^1(\Omega)\) such that

\[
h_{0,n} \to h_0 \quad \text{in } L^2(\Omega) \text{ strongly},
\]  
(52)

and, finally, let \(\{z_{0,n}\}\) be a sequence in \(L^2(\Omega)\) such that

\[
z_{0,n} \to z_0 \quad \text{in } L^2(\Omega) \text{ strongly}.
\]  
(53)

For the sediment equation, we can easily observe from (36) that \(\nabla \cdot u\) belongs to \(L^2(\Omega)\), and therefore \(\partial_t z\) also belongs to \(L^2(\Omega)\). Let us show that \(z(t = 0) = z_0\).
By using triangular inequality, we have
\[ \| z(t) - z_0 \| \leq \| z(t) - z_n(t) \| + \| z_n(t) - z_{0,n} \| + \| z_{0,n} - z_0 \|. \] (54)

From (42), we can easily show that
\[ \| z(t) - z_n(t) \| \to 0 \quad \text{as} \quad n \to \infty. \] (55)

The second term in the right-hand side of (54) goes to zero when \( t \to 0 \), according to the definition of the initial conditions of the sequence \((z_n)_n\). From (53), we deduce that the last term in the right-hand side of (54) goes to zero when \( n \to \infty \). Thus \( z(t = 0) = z_0 \). By using estimate (34), we deduce that \( \nabla \cdot (u h) \) belongs to \( L^2(0, T; H^{-1}(\Omega)) \), and so \( \partial_t h \) as well. We also have \( h(t = 0) = h_0 \), and we can pass to the limit in the momentum equation and obtain \( u(t = 0) = u_0 \) using the same steps as previously.

This concludes the proof that \( (u, h, z) \) is a solution of the weak problem (\( \mathcal{V} \)). \( \square \)

5. The discrete problem: convergence of a discontinuous–continuous Galerkin finite element scheme

In this section, we employ a continuous–discontinuous Galerkin finite element scheme [6] to numerically solve system (\( \mathcal{P} \)).

5.1. Notation and function space properties

Let \( \{ \tau_h \}_{h>0} \) be a family of finite element partitions of \( \Omega \) such that no element \( \Omega_e \) crosses \( \partial \Omega \).

Denote by \( l_e \) an element diameter of each element \( \Omega_e \), with \( l \) being the maximal element diameter. We assume that each element \( \Omega_e \) is affinely equivalent to one of several reference elements [9]. Denote by \( \mathcal{P}^k (\Omega_e) \) the space of complete polynomials of degree \( k \geq 1 \), defined on \( \Omega_e \). For any function \( v \in H^1(\Omega_e) \), we denote as in [6] its trace on interior edges, \( \gamma_i \), by \( v^\pm \):
\[ v^-(x) = \lim_{s \to 0^-} v(x + sn_i), \quad v^+(x) = \lim_{s \to 0^+} v(x + sn_i). \] (56) (57)

Now, we define
\[ \mathfrak{n} = \frac{1}{2} (v^+ + v^-), \quad [v] = v^- - v^+, \] (58) (59)

where \( x \in \gamma_i \) and \( n_i \) denotes a fixed unit outward vector. Define \( \sum_{i} \) as summation over all interior element edges \( \gamma_i \), and, for each \( g \in L^2(\Omega_e) \), we define \( \| g \|^2_{\Omega_e} = \sum_{i} \| g \|^2_{\Omega_e \gamma_i} \).

In our analysis, we will use the following well-known trace theorem [9].

**Theorem 5.1.** Suppose that \( \Omega_e \) has a Lipschitz boundary. Then, there is a constant \( K_e^t \) such that
\[ \| v \|_{L^2(\partial \Omega_e)} \leq K_e^t \| v \|_{L^2(\Omega_e)}^{1/2} \| v \|_{H^1(\Omega_e)}^{1/2}. \] (60)

Define
\[ K^t = \max_e K_e^t. \] (61)

5.2. The discontinuous–continuous Galerkin formulation

In this section, we will adopt a discontinuous–continuous Galerkin formulation. Indeed, we will discretize the water and sediment continuity equations, i.e., (\( \mathcal{P} \)) and (\( \mathcal{P} \)), by a discontinuous Galerkin (DG) method and the momentum equations by a standard Galerkin finite element method. Multiplying (\( \mathcal{P} \)) and (\( \mathcal{P} \)) by arbitrary smooth test functions \( v \in H^1(\Omega_e) \) and integrating by parts over each element \( \Omega_e \), we obtain
\[ (\partial_t h, v)_{\Omega_e} - (u, \nabla v)_{\Omega_e} + (h u \cdot n_e, v)_{\partial \Omega_e} = 0, \] (62)
\[ (\partial_t z, v)_{\Omega_e} - \sigma (q_e, \nabla v)_{\Omega_e} + \sigma (q_e \cdot n_e, v)_{\partial \Omega_e} = 0. \] (63)

On each \( \Omega_e \), we approximate \( h \) and \( z \) in a space \( \mathcal{P}^k (\Omega_e) \), where \( \mathcal{P}^k (\Omega_e) \subset \mathcal{P}^k (\Omega_e) \), and we define
\[ V_h = \left\{ v : \Omega \to \mathbb{R} : v|_{\Omega_e} \in \mathcal{P}^k (\Omega_e) \right\}. \] (64)
Denote by \( V^c_h \) the space of continuous piecewise polynomials of degree \( k \). That is,
\[
V^c_h = V_h \cap c^0(\Omega) \neq \emptyset.
\]

By multiplying (74) by \( w \in (H^1_0(\Omega))^2 \) and integrating by parts over the domain \( \Omega \), we obtain
\[
(\partial_t u, w)\Omega + (g \nabla (h + z), w)\Omega = \sum_i \langle g [h + z]_\gamma, w \cdot n_i \rangle_{\gamma_i} + (A \nabla u, \nabla w)\Omega + ((u \cdot \nabla) u, w)\Omega = 0. 
\]

Note that the stabilization term \( \sum_i \langle g [h + z]_\gamma, w \cdot n_i \rangle_{\gamma_i} \) is actually zero since we are assuming that our true solution is sufficiently smooth as to be continuous.

Let us approximate \( u \) in the finite-dimensional subspace \( W_h \subset (H^1(\Omega))^2 \cap \{ u : u = \hat{u} \text{ on } \partial \Omega \} \) such that each component of \( u \) is in \( V^c_h \), and denote by \( W_{0,h} \) the corresponding subspace of \( (H^1_0(\Omega))^2 \). Approximate \( h(\cdot, t) \) by \( H(\cdot, t) \in V_h \) and \( u(\cdot, t) \) by \( U(\cdot, t) \in W_h \), and let the value of \( h \) across inner element boundaries be approximated by a simple upwinding scheme denoted by \( H^1 : \)
\[
h \approx H^1 = \begin{cases} H^-, & U \cdot n_i > 0 \text{ on } \gamma_i, \\ H^+, & U \cdot n_i \leq 0 \end{cases}
\]

At \( t = 0 \), define \( H(\cdot, t) = H_0 \in V_h \), \( Z(\cdot, t) = Z_0 \in V_h \) and \( U(\cdot, t) = U_0 \in W_h \) by
\[
(H_0 - h_0, v)_\Omega = 0, \quad \forall v \in V_h, \\
(Z_0 - z_0, v)_\Omega = 0, \quad \forall v \in V_h, \\
(U_0 - u_0, w)_\Omega = 0, \quad \forall w \in W_{0,h}.
\]

The discrete weak formulation is the following.

For each \( t > 0 \), find \( (H, Z, U) \in V^c_h \times W_h \) satisfying, \( \forall v \in V_h \) and \( \forall w \in W_{0,h}, \)
\[
\sum_c (\partial_c H, v)_{\Omega_c} - \sum_c (UH, \nabla v)_{\Omega_c} + \sum_i \langle [H^1 U \cdot n_i, [v]] \rangle_{\gamma_i} + \langle [H^1 \hat{u} \cdot n_i, v] \rangle_{\partial \Omega_1} = - \langle [\hat{u} \cdot n_i, v] \rangle_{\partial \Omega_1}, \\
\sum_c (\partial_c Z, v)_{\Omega_c} - \sum_c (\sigma C U, \nabla v)_{\Omega_c} + \sum_i \langle [\sigma C U \cdot n_i, [v]] \rangle_{\gamma_i} + \langle [\sigma C \hat{u} \cdot n_i, v] \rangle_{\partial \Omega_1} = 0, \\
(\partial_t U, w)_{\Omega} + (g \nabla (H + Z), w)_{\Omega} - \sum_i \langle g [H + Z]_\gamma, w \cdot n_i \rangle_{\gamma_i} + (A \nabla U, \nabla w)_{\Omega} + ((U \cdot \nabla) U, w)_{\Omega} = 0.
\]

5.2.1. An a priori error estimate

We define \( h^l(\cdot, t) \in V^c_h \) to be the continuous interpolant of \( h \). Moreover, we define the parabolic injection \( \Pi u(\cdot, t) \in W_h \) such that
\[
(\partial_c (\Pi u - u), w)_{\Omega} + (A \nabla (\Pi u - u), \nabla w)_{\Omega} = 0, \quad \forall w \in W_{0,h}(\Omega),
\]
with \( \Pi u(\cdot, t) \) equal to the \( L^2 \) projection of \( u_0 \) into \( W_h \); that is,
\[
(\Pi u(\cdot, 0) - u_0, w)_{\Omega} = 0, \quad \forall w \in W_{0,h}(\Omega).
\]

Let us recall a standard inverse inequality, valid for continuous (and discontinuous) polynomials on quasi-uniform triangulations [9], which we will use in our analysis.

**Theorem 5.2.** Let \( v \in V_h \). Then
\[
\| v \|_{H^1(\Omega)} \leq K' T^{-1} \| v \|_{L^2(\Omega)},
\]
where \( K' \) is independent of \( l \).

Our estimates will rely on certain smoothness of the solutions. We assume that the following constants are finite:
\[
K_h = \int_0^T \left[ \| \partial_t h \|_{H^{k+1}(\Omega)} + \| h \|_{H^{k+1}(\Omega)}^2 \right] dt + \| h_0 \|_{H^{k}(\Omega)}^2, \\
K_z = \int_0^T \left[ \| \partial_t z \|_{H^{k+1}(\Omega)} + \| z \|_{H^{k+1}(\Omega)}^2 \right] dt + \| z_0 \|_{H^{k}(\Omega)}^2, \\
K_u = \int_0^T \left[ \| \partial_t u \|_{H^{k}(\Omega)} + \| u \|_{H^{k+1}(\Omega)}^2 \right] dt + \| u_0 \|_{H^{k}(\Omega)}^2.
\]
\[ K^*_1 = \|h_1\|_{L^\infty(0,T;W^2_0(\Omega))}, \]  
\[ K^*_2 = \|z_1\|_{L^\infty(0,T;W^1_0(\Omega))}, \]  
\[ K^*_3 = \|u\|_{L^\infty(0,T;W^1_0(\Omega))}. \]

Define
\[ e_h = H - h_1, \quad e_z = Z - z_1, \quad e_u = U - \Pi u, \]
\[ \theta_h = h - h_1, \quad \theta_z = z - z_1, \quad \theta_u = u - \Pi u. \]

Our main result is the following.

**Theorem 5.3.** For \(u, h, z\) sufficiently smooth, scheme (71)–(73) satisfies the error estimate
\[ \| (e_h, e_z, e_u) \| \leq K^*, \]  
where
\[ 2\| (e_h, e_z, e_u) \|^2 = 2\| (e_h, e_u) \|^2 + \| e_z(T) \|^2, \]
with
\[ 2\| (e_h, e_u) \|^2 = \| e_h(T) \|^2 + \| e_u(T) \|^2 + \frac{1}{8} A \int_0^T \| \nabla e_u \|^2 dt + R_h, \]
where
\[ R_h = \int_0^T \sum_i \langle [U \cdot n_i], [e_h]^2 \rangle d\gamma + \int_0^T \sum_i \langle [\tilde{u} \cdot n], e_h \rangle_{\partial \Omega_i} dt + \int_0^T \sum_i \langle U \cdot n, e_h^2 \rangle_{\partial \Omega_i} dt, \]
and \(K^*\) is a constant independent of \(l\) and \(k\) but dependent on \(g, A, K^1, K_h, K_z, K_u, K^*_1, K^*_2, K^*_3, K^*_u\).

**Proof.** Standard approximation results for \(h_1, z_1,\) and \(\Pi u\) give
\[ \int_0^T \left( \| \partial_t \theta_h \|^2 + \| \theta_h \|_{H^1(\Omega)}^2 + \frac{1}{2} \| \theta_h(0) \|^2 \right) dt \leq K(K_h)^2, \]
\[ \int_0^T \left( \| \partial_t \theta_z \|^2 + \| \theta_z \|_{H^1(\Omega)}^2 + \frac{1}{2} \| \theta_z(0) \|^2 \right) dt \leq K(K_z)^2, \]
\[ \int_0^T \left( \| \theta_u \|_{H^1(\Gamma)}^2 + \frac{1}{2} \| \theta_u \|^2 \right) dt \leq K(A, K_u)^2. \]

Subtract the weak formulation from the corresponding discrete formulations (71)–(73), and integrate in time from 0 to \(T\). Incorporate \(\Pi u\) and \(h_1\) and \(z_1\) and take \(v = e_h, v = e_z\) and \(w = e_u\) in (71)–(73), respectively, to obtain
\[ \int_0^T \sum_e \langle \partial_t e_h, e_h \rangle_{\Omega_e} dt - \int_0^T \sum_e \langle U e_h, \nabla e_h \rangle_{\Omega_e} dt + \int_0^T \sum_i \langle e_h U \cdot n_i, [e_h] \rangle_{\gamma_i} dt \]
\[ + \int_0^T \langle e_h \tilde{u} \cdot n, e_h \rangle_{\partial \Omega_0} dt + \int_0^T \langle \partial_t e_u, e_u \rangle_{\Omega_e} dt + \int_0^T \sum_e \langle g \nabla e_u, e_u \rangle_{\Omega_e} dt \]
\[ - \int_0^T \sum_i \langle g[e_h], e_u \cdot n_i \rangle_{\gamma_i} dt + \int_0^T \langle A \nabla e_u, \nabla e_u \rangle_{\Omega_e} dt \]
\[ = \int_0^T \sum_e \langle \partial_t \theta_h, e_h \rangle_{\Omega_e} dt - \int_0^T \sum_e \langle u h - h U, \nabla e_h \rangle_{\Omega_e} dt + \int_0^T \sum_i \langle [h u \cdot n_i - h_1 U \cdot n_i, [e_h] \rangle_{\gamma_i} dt \]
\[ + \int_0^T \langle \partial_t \tilde{u} \cdot n, e_h \rangle_{\partial \Omega_0} dt + \int_0^T \sum_e \langle g \nabla \theta_h, e_u \rangle_{\Omega_e} dt - \int_0^T \sum_e \langle g \nabla e_z, e_u \rangle_{\Omega_e} dt. \]
\[
+ \int_0^T \sum_e (g[e_2], e_u \cdot n_e)_{\Omega_e} dt - \int_0^T \sum_e ((U \cdot \nabla)U - (u \cdot \nabla)u, e_u)_{\Omega_e} (\text{XV})
\]

and
\[
\int_0^T \sum_e (\partial_t e_2, e_2)_{\Omega_e} dt - \int_0^T \sum_e (\sigma C_i \nabla e_2, e_u)_{\Omega_e} dt + \int_0^T \sum_i (\sigma C_i [e_2], e_u \cdot n_i)_{\gamma_i} dt
\]

\[
= \int_0^T \sum_e (\partial_t \theta, e_2)_{\Omega_e} dt - \int_0^T \sum_e (\sigma C_i \nabla e_2, \theta_u)_{\Omega_e} dt + \int_0^T \sum_i (\sigma C_i [e_2], \theta_u \cdot n_i)_{\gamma_i} dt. (\text{IX})
\]

The terms in (88), except the three last ones, have already been estimated in [6].

Let us estimate the terms (XIV) and (XV) in (88).

\[
- \int_0^T \sum_e (g \nabla e_2, e_u)_{\Omega_e} dt + \int_0^T \sum_i \langle g[e_2], e_u \cdot n_i \rangle_{\gamma_i} dt = \int_0^T \sum_e (ge_2, \nabla \cdot e_u)_{\Omega_e}.
\]

By using Schwartz and Young inequalities, we obtain
\[
\int_0^T \sum_e (ge_2, \nabla \cdot e_u)_{\Omega_e} \leq \int_0^T g \|e_2\| \|\nabla e_u\| dt
\]
\[
\leq \frac{2g^2}{A} \int_0^T \|e_2\|^2 dt + \frac{1}{8} A \int_0^T \|\nabla e_u\|^2 dt. (90)
\]

Let us estimate the last term in (88).

\[
- \int_0^T \sum_e ((U \cdot \nabla)U - (u \cdot \nabla)u, e_u)_{\Omega_e} = - \int_0^T \sum_e ((U \cdot \nabla)e_u, e_u)_{\Omega_e} + \int_0^T \sum_e ((U \cdot \nabla)\theta_u, e_u)_{\Omega_e}
\]
\[
- \int_0^T \sum_e ((e_u \cdot \nabla)u, e_u)_{\Omega_e} + \int_0^T \sum_e ((\theta_u \cdot \nabla)u, e_u)_{\Omega_e}. (91)
\]

Before estimating the terms in the right-hand side of (91), note that, since \(U \in W_h\), then, for each \(t \in ]0, T]\), we have \(\|U(t)\|_{L^\infty(\Omega)}^2\) is bounded. Therefore, by using the regularity of the basis of \(W_h\), one can assume that the coefficients of \(\|U(t)\|_{L^\infty(\Omega)}^2\) are bounded in \(]0, T]\). Thus, there exists \(N > 0\), independent of \(t\), such that
\[
\|U\|_{C([0,T];L^\infty(\Omega))} \leq N. (92)
\]

The first term in the right-hand side of (91) can be estimated as
\[
- \int_0^T \sum_e ((U \cdot \nabla)e_u, e_u)_{\Omega_e} \leq \frac{1}{2\epsilon} \int_0^T \|U\|_{L^\infty(\Omega)}^2 \|e_u\|^2 + \frac{\epsilon}{2} \int_0^T \|\nabla e_u\|^2
\]
\[
\leq \frac{N^2}{2\epsilon} \int_0^T \|e_u\|^2 + \frac{\epsilon}{2} \int_0^T \|\nabla e_u\|^2. (93)
\]

The second term in the right-hand side of (91) can be estimated as
\[
\int_0^T \sum_e ((U \cdot \nabla)\theta_u, e_u)_{\Omega_e} \leq \frac{1}{2\epsilon_1} \int_0^T \|U\|_{L^\infty(\Omega)}^2 \|e_u\|^2 + \frac{\epsilon_1}{2} \int_0^T \|\nabla \theta_u\|^2
\]
\[
\leq \frac{N^2}{2\epsilon_1} \int_0^T \|e_u\|^2 + \frac{\epsilon_1}{2} K(A, K_u)\|e_u\|^{2k}. (94)
\]

The third term in the right-hand side of (91) can be estimated as
\[
- \int_0^T \sum_e ((e_u \cdot \nabla)u, e_u)_{\Omega_e} \leq 2 \int_0^T \|\nabla u\|_{L^\infty(\Omega)} \|e_u\|^2
\]
\[
\leq 2K^* \int_0^T \|e_u\|^2. (95)
\]
The last term in the right-hand side of (91) can be estimated as

\[
\int_0^T \sum_e (\varepsilon \cdot \nabla u, e_u)_{\Omega_e} \leq \frac{1}{\varepsilon^2} \int_0^T \|\nabla u\|^2_{L^\infty(\Omega)} \|e_u\|^2 + \varepsilon_2 \int_0^T \|\nabla \varepsilon_0\|^2 \\
\leq \frac{(K_u^*)^2}{\varepsilon_2} \int_0^T \|e_u\|^2 + \varepsilon_2 K(A, K_u)l^2.
\]

By using (90)–(96) and the estimation results in [6], we show that

\[
\|e_u(T)\|_{L^2}^2 + \|e_0(T)\|^2 + \frac{1}{2} \int_0^T \|\nabla e_u\|^2 \, dt + R_0 \leq K_0 l^2 K + K_1 \int_0^T (\|e_u\|^2 + \|e_0\|^2 + \|e_2\|^2) \, dt.
\]

In order to estimate \(\|e_2(T)\|\), let us now estimate each term of (89). The first term of (89) can be estimated as follows.

\[
\int_0^T \sum_e (\partial_t e_2, e_2)_{\Omega_e} \, dt = \frac{1}{2} \|e_2(T)\|^2 - \frac{1}{2} \|e_2(0)\|^2 \\
\geq \frac{1}{2} \|e_2(T)\|^2 - K\|e_2\|^2.
\]

By integrating by parts, the second and the third terms of (89) can be rewritten as

\[
- \int_0^T \sum_e (\sigma C_\gamma \nabla e_2, e_u)_{\Omega_e} \, dt + \int_0^T \sum_i (\sigma C_\gamma \langle e_2, e_u \rangle_{\gamma_i})_\Omega \, dt \\
= \int_0^T \sum_e (\sigma C_\gamma \nabla \cdot e_2, e_u)_{\Omega_e} \, dt - \int_0^T (\sigma C_\gamma \langle e_2, e_u \rangle_{\gamma\partial\Omega}) \, dt.
\]

By using Schwartz and Young inequalities, the first term in the right-hand side of (99) can be estimated as

\[
\int_0^T \sum_e (\sigma C_\gamma \nabla \cdot e_2, e_u)_{\Omega_e} \, dt \leq \frac{1}{4} A \int_0^T \|\nabla e_u\|^2 \, dt + K_4 \int_0^T \|e_2\|^2 \, dt.
\]

Since \(\Pi u\) and \(U\) are in \(V_h\), that is

\[
\Pi u = U = \mathring{u} \quad \text{in} \quad \partial \Omega,
\]

then \(e_u = 0\) in \(\partial \Omega\). Consequently, the last term in the right-hand side of (99) vanishes. Therefore,

\[
\|I - z + III - z \leq \frac{1}{4} A \int_0^T \|\nabla e_u\|^2 \, dt + K_4 \int_0^T \|e_2\|^2 \, dt.
\]

The first term in the right-hand side of (89) is estimated by using Schwartz and Young inequalities:

\[
IV - z = \int_0^T \sum_e (\partial_t \theta_2, e_2)_{\Omega_e} \, dt \leq \frac{1}{2} \int_0^T (\|\partial_t \theta_2\|^2 + \|e_2\|^2) \, dt.
\]

By using estimation (86), we obtain

\[
IV - z \leq K(K_2^*) l^2 + \frac{1}{2} \int_0^T \|e_2\|^2 \, dt.
\]

By integrating the second term in the right-hand side of (89) by parts and combining it with the last term of (89), we arrive at

\[
- \int_0^T \sum_e (\sigma C_\gamma \nabla e_2, \theta_u)_{\Omega_e} \, dt + \int_0^T \sum_i (\sigma C_\gamma \langle e_2, \theta_u \rangle_{\gamma_i})_\Omega \, dt \\
- \int_0^T (\sigma C_\gamma e_2, \nabla \cdot \theta_u)_{\Omega_e} \, dt - \int_0^T (\sigma C_\gamma \langle e_2, \theta_u \rangle_{\gamma\partial\Omega}) \, dt.
\]
The first term in the right-hand side of (105) can be estimated by using Schwartz and Young inequalities as

$$
\int_0^T (\sigma C e_z, \nabla \cdot \theta_u) \Omega \, dt \leq K(\sigma, C_z) \int_0^T \| \nabla \cdot \theta_u \| \| e_z \| \, dt \\
\leq \frac{1}{2} K(\sigma, C_z) \int_0^T (\| \theta_u \|_{H^2(\Omega)}^2 + \| e_z \|^2) \, dt \\
\leq \frac{1}{2} K(\sigma, C_z, A, K_u)^{2k} + \frac{1}{2} K(\sigma, C_z) \int_0^T \| e_z \|^2. 
$$

(106)

By using Theorem 5.1, the last term in the right-hand side of (105) is estimated as

$$
- \int_0^T (\sigma C e_z, \theta_u \cdot \mathbf{n})_{\partial \Omega} \, dt \leq K(\sigma, C_z) \int_0^T r^{\frac{1}{2}} \| \theta_u \|_{\partial \Omega}^{\frac{3}{2}} \| e_z \|_{\partial \Omega} \, dt \\
\leq \frac{1}{2} K(\sigma, C_z) \int_0^T (r^{-1} \| \theta_u \|_{\partial \Omega}^2 + l \| e_z \|^2_{\partial \Omega}) \, dt \\
\leq K(K^1, \sigma, C_z) \int_0^T (r^{-2} \| \theta_u \|^2 + \| e_z \|^2_{H^1(\Omega)}) \, dt \\
\leq \frac{1}{2} K(K^1, \sigma, C_z) \int_0^T (\| e_z \|^2 + l^2 \| e_z \|^2_{H^1(\Omega)}) \, dt \\
+ \frac{1}{2} K(K^1, \sigma, C_z) \int_0^T (\| e_z \|^2 + l^2 \| e_z \|^2_{H^1(\Omega)}) \, dt.
$$

(107)

Estimation (87) gives

$$
\frac{1}{2} K(K^1, \sigma, C_z) \int_0^T (r^{-2} \| \theta_u \|^2 + \| e_z \|^2_{H^1(\Omega)}) \, dt \leq K(K^1, \sigma, C_z, A, K_u)^{2k}. 
$$

(108)

Theorem 5.2 implies that

$$
\int_0^T (\| e_z \|^2 + l^2 \| e_z \|^2_{H^1(\Omega)}) \, dt \leq K(K^1, K^1) \int_0^T \| e_z \|^2 \, dt.
$$

(109)

Therefore,

$$
V - z + \psi - z \leq K_5 t^{2k} + K_6 \int_0^T \| e_z \|^2 \, dt.
$$

(110)

By combining the estimations (98), (102), (104) and (110), we have

$$
\| e_z(T) \|^2 \leq K_7 t^{2k} + K_9 A \int_0^T \| e_z \|^2 \, dt + \frac{1}{4} \int_0^T \| \nabla e_u \|^2 \, dt.
$$

(111)

Finally, by combining (97) and (111), we obtain

$$
\| e_u(T) \|^2 + \| e_b(T) \|^2 + \| e_z(T) \|^2 + \frac{1}{4} \int_0^T \| \nabla e_u \|^2 \, dt + R_n \\
\leq K_{10} t^{2k} + K_{13} \int_0^T (\| e_u \|^2 + \| e_b \|^2 + \| e_z \|^2) \, dt.
$$

(112)

Let us recall the following Gronwall lemma. □

**Lemma 5.4.** Let φ, ψ, and y be three continuous functions in a segment [0, b], with positive real values and verifying the following inequality:

$$
y(T) \leq \phi(T) + \int_0^T \psi(t) y(t) \, dt \quad \forall T \in [0, b].
$$

(113)

Then we have

$$
y(T) \leq \phi(T) + \int_0^T \phi(t) \psi(t) \exp \left( \int_t^T \psi(s) \, ds \right) \, dt \quad \forall T \in [0, b].
$$

(114)
By choosing $\varepsilon = \frac{1}{4}$ in (112), and by setting
\[ y(T) = \| e_n(T) \|^2 + \| e_h(T) \|^2 + \| e_z(T) \|^2 + \frac{1}{8} A \int_0^T \| \nabla e_u \|^2_{L^2} dt + R_h, \]
\[ \phi(T) = K_{10} l^{2k}, \]
\[ \psi(T) = K_{11}, \]
relation (112)
\[ y(T) \leq \phi(T) + \int_0^T \psi(t) y(t) \, dt \quad \forall T \in [0, b]. \] (116)
Then by using the Gronwall lemma, Lemma 5.4, we obtain
\[ y(T) \leq \phi(T) + \int_0^T \phi(t) \psi(t) \exp \left( \int_t^T \psi(s) \, ds \right) \, dt \quad \forall T \in [0, b]. \] (117)
By replacing $\phi, \psi, \text{ and } y$ by their expressions in (115), we obtain
\[ \| e_n(T) \|^2 + \| e_h(T) \|^2 + \| e_z(T) \|^2 + \frac{1}{8} A \int_0^T \| \nabla e_u \|^2_{L^2} dt + R_h \]
\[ \leq K_{10} l^{2k} + K_{10} K_{11} l^{2k} \int_0^T \exp \left( \int_t^T K_{11} ds \right) \, dt \quad \forall T \in [0, b]. \] (118)
Thus
\[ \| e_n(T) \|^2 + \| e_h(T) \|^2 + \| e_z(T) \|^2 + \frac{1}{8} A \int_0^T \| \nabla e_u \|^2_{L^2} dt + R_h \leq K_{10} l^{2k} + K_{10} K_{11} l^{2k} \exp(K_{11} b). \] (119)
Consequently, we have
\[ \| (e_n, e_z, e_u) \| \leq K^k. \] (120)

6. Conclusion

We have considered a two-dimensional shallow water system over movable beds. First, we proved the existence of solutions of the continuous problem. Indeed, we first built the corresponding finite-dimensional problem and showed, by using some compactness results, that the solution sequence $(u_n, h_n, z_n)$ is convergent and that its limit $(u, h, z)$ is a solution of the continuous problem. We also investigated the smoothness of this continuous solution. We showed that this solution satisfies $u$ in $C^0([0, T]) \cap L^2([0, T]) \cap L^2([0, T]; \mathbf{V}) \cap L^2([0, T]; (e^{\partial(\mathbf{Q})})^2)$; $h$ in $C^1([0, T]; \mathbf{Q})$ and $z$ in $L^\infty([0, T]; L^2(\Omega))$.

We also considered a continuous–discontinuous Galerkin scheme for the numerical solution of the system and performed an error analysis for the scheme. Each component of the solution was approximated in a finite-dimensional subspace with elements belonging piecewisely to $\mathcal{P}^k$. Then, by choosing $l$ as the maximal element diameter of the element partition of $I$, we showed that the error norm is bounded and that it is of $O(l^k)$.

Acknowledgments

The authors thank an anonymous reviewer for the careful review of the paper. The research was supported by Natural Sciences and Engineering Council of Canada.

References