Numerical modeling of submarine turbidity currents over erodible beds using unstructured grids

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A B S T R A C T

Second-order central-upwind schemes proposed by Bryson et al. (2011) for the Saint–Venant system have
two very attractive properties: well-balanced and positivity preserving, which are originally designed for
constant fluid density and fixed beds in Bryson et al. (2011). For the turbidity current system with vari-
dable density over erodible beds, such desired properties can be obtained by developing a well-balanced
and positivity preserving central-upwind scheme following the ideas in Bryson et al. (2011). To this end,
in this paper, a coupled numerical model for two-dimensional depth-averaged turbidity current system
over erodible beds is developed using finite volume method on triangular grids. The proposed numerical
model is second-order accurate in space using piecewise linear reconstruction and third-order accurate
in time using a strong stability preserving Runge–Kutta method. Applying the central-upwind method to
estimate numerical fluxes through cell interfaces, the model can successfully deal with sharp gradients in
turbidity flows. The developed numerical model can preserve the well-balanced property over irregular
bottom, guarantee the non-negative turbidity current depth over erodible beds, and preserving the pos-
itivity of suspended sediment. These features of the developed numerical model and its robustness and
accuracy are demonstrated in several numerical tests.

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1. Introduction

Turbidity current is a type of underwater density current driven
by the negative buoyancy forces due to the suspended sediment.
Oceanic turbidity currents can be caused by tectonic disturb-
ances of the sea floor, and the slumping of sediment that has piled up
at the top of convergent plate margins, continental slopes and subma-
rine canyons. Their profound impacts on the continental shelves,
sea floor and ocean environment have attracted intensive research
interests.

A large number of laboratory tests have been conducted to
study the turbidity currents, see Parker (1982), Garcia et al. (1985),
Parker et al. (1987), Bonnecaze et al. (1995), Garcia (1993), Parker
et al. (1986), Alexander and Mulder (2002), Baas et al. (2004),
Cantelli et al. (2011), Hallworth and Huppert (1998), Sequeiros
et al. (2009), Yu et al. (2006), Janocko et al. (2013), Motanated
and Tice (2016), Chowdhury and Testik (2011) and Fedele and Gar-
cia (2009). However, laboratory studies are usually constrained by
their small scale and can only be roughly applied to large-scale
geophysical events. Only a few field observations of turbidity cur-
rents are reported in the literature due to the unpredictability and
destructive nature of such events, see Inman et al. (1976), Hay
(1987) and Normark (1989). Due to the aforementioned constrains
of experimental and observational studies, numerical investigations
based on theoretical and physical models have attracted significant
attention because of their affordable computational cost and avail-
able details of time-dependent flow structure. Three-dimensional
(3-D) models can resolve the vertical structure of turbidity current,
particularly for the current front that may be considerably non-
uniform in vertical. Several 3-D numerical investigation of density
currents have been performed, see Huang et al. (2005), Kasem
and Imran (2004), Necker et al. (2002), El-Gawad et al. (2012) and
Ooi et al. (2015). However, the computational costs of such 3-D
numerical models are very expensive, especially when applied to
large scale cases. In addition, the physics of turbulent density cur-
rents has not been well understood; this also limits the applica-
tions of the 3-D turbidity current models. An alternative way to
numerically investigate the submarine turbidity current is to use
the depth-averaged two dimensional (2-D) numerical models
which are attractive due to the fact that such 2-D models are well-
understood, of lower computational cost and easy to use. One-
layer 2-D depth averaged models have been widely studied. Parker
et al. (1986) derived the depth-averaged governing equations for 2-
D turbidity currents which provides the basic model for numerical

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studies. Choi (1998) developed a 2-D numerical model for turbidity currents using a dissipative-Galerkin finite element method. A deforming grid generation algorithm was used to track the current front. However, such techniques may become very expensive if the solution in the far-field is of interest because a large number of elements are required to produce accurate solutions. This prevents the extension of such a model to large-scale studies. Bradford and Katopodes (1999) developed a 2-D finite-volume method for turbid underflows considering non-uniform sediment. Roe scheme is used to estimate fluxes through cell interfaces. Their model didn’t include the friction of ambient water which may lead to overestimation of propagating speed. Imran et al. (1998) developed a 2-D numerical model describing the formation of a submarine fan by a spreading turbidity current using finite difference method. In those models, the bed evolves in response to the exchange of suspended sediment in the turbidity current. However, the momentum and mass exchange along with sediment exchange are not considered in their studies. To fully consider the feedback impacts of bed deformation, Hu and Cao (2009) have modified the governing equations for turbidity current and extended it to 2-D cases (Hu et al., 2012a, 2012b) based on finite volume method. Those modified equations are also adopted in the current study. One desired property of such numerical models for slender flows, especially for turbidity currents, is positivity preserving. Although applying Courant-Friedrichs-Lewy (CFL) condition is sufficient to guarantee the numerical stability of finite volume methods, using the time step size satisfying CFL condition can not guarantee the non-negativity of current depth and volumetric concentration of sediment. None of the above-mentioned numerical models for turbidity currents incorporates the positivity preserving property. In applications, the current depth and volumetric concentration may be forced to be zero if the computed value is negative. This leads to a loss of certain amount of mass and momentum. In this study, the positivity preserving properties for both depth and concentration of turbidity current are guaranteed using the developed numerical model.

The developed numerical model in the current study is an extension of the well-balanced positivity preserving central-upwind methods developed by Bryson et al. (2011). However, the well-balanced and positivity preserving techniques in Bryson et al. (2011) were originally designed for Saint-Venant system with constant fluid density and fixed bed. Consequently, they can not be directly applied to the studied turbidity current system. To overcome this, the well-balanced and positivity techniques in Bryson et al. (2011) are modified in our proposed schemes, and a numerical model with such desired properties is developed.

In this paper, a numerical model based on the finite volume method, is developed to solve 2-D coupled depth-averaged turbidity current system consisting of sediment transport, hydrodynamic and morphodynamic models. The resulting hyperbolic system of balance laws consists of five coupled equations. The central-upwind method (Kurganov and Tadmor, 2000; Kurganov et al., 2001; Kurganov and Petrova, 2005, 2007; Kurganov and Levy, 2002) is adopted to estimate the numerical fluxes through cell interfaces. In order to guarantee the well-balanced property, special discretization for bed-slope source term is developed in this paper. Using non-negative reconstruction for bed, desingularization of point values and special treatment at wet-dry fronts, the model can deal with wetting and drying over complex geometry while obtaining the well balanced property at wet-dry boundaries. The positivity preserving property of the proposed model is proved.

It is noted that, due to the existence of ambient water, there is no true dry bed involved. The wetting-drying process mentioned in the current paper refers in particular to the wetting and drying process of turbidity current in the ambient water.

This paper is organized as follows. In Section 2, the governing equations and closures are presented, for which in Section 3 the well-balanced and positivity preserving techniques for turbidity current system are developed. Several numerical tests are conducted and presented in Section 4. Some concluding remarks complete the study in Section 5.

2. Governing equations

The governing equations for modeling the two-dimensional spreading of turbidity currents over erodible bed are conservation equations for fluid, sediment and bed material, respectively. A detailed derivation of the governing equations for 2-D turbidity currents can be found in Parker et al. (1986). In this study, the layer-averaged 2-D governing equations with non-equilibrium sediment transport adopted in Hu et al. (2012b) are used:

\[ h_i + (hu)_x + (hv)_y = \omega_w |V| + (E - D)/(1 - p). \]

\[ (hu)_x + \left( hu^2 + \frac{g}{2} h^2 \right)_x + (huV)_y = -g h Z_x + (1 + r_w) u_i^2 - \frac{u(E - D)(\rho_0 - \rho)}{\rho(1 - p)} - \frac{\rho_w - \rho}{\rho} \omega_w |V|, \]

\[ (hv)_y + (huv)_x + \left( hv^2 + \frac{g^2}{2} h^2 \right)_y = -g h Z_y + (1 + r_v) v_i^2 - \frac{v(E - D)(\rho_0 - \rho)}{\rho(1 - p)} - \frac{\rho_w - \rho}{\rho} \omega_v |V|, \]

\[ (hc)_t + (huc)_x + (hvc)_y = E - D. \]

The bed evolution is computed by using the Exner equation for the conservation of bed sediment, which takes the form:

\[ Z_t = -\frac{E - D}{1 - p}. \]

In Eqs. (1)–(5), \( t \) is time, \( x \) and \( y \) are horizontal coordinates, \( h = h(x, y, t) \) is the thickness of the turbidity current layer, \( u = u(x, y, t) \) and \( v = v(x, y, t) \) are the \( x \) - and \( y \)-components of the layer-averaged velocities, respectively, \( c = c(x, y, t) \) is the layer-averaged volumetric sediment concentration, \( Z = Z(x, y, t) \) is the bed elevation, \( |V| = \sqrt{u^2 + v^2} \) is the total velocity, \( g \) is the gravitational acceleration, \( g = R gc \) is the submerged gravitational acceleration, \( R = (\rho_s - \rho_w)/\rho \) is submerged specific gravity, \( \rho_w \) and \( \rho_s \) are the densities of water and sediment, respectively, \( \rho = \rho_w (1 - c) + \rho_s c \) is the density of the water-sediment mixture, \( p \) is the bed porosity, \( \rho_0 = \rho_w p + \rho_s (1 - p) \) is the density of the saturated bed, \( u_i = \sqrt{C_0 u |V|} \) and \( v_i = \sqrt{C_0 v |V|} \) are the components of the friction source in the \( x \) - and \( y \)-directions, respectively, \( C_0 \) is the bed drag coefficient with a typical range 0.002-0.06 (Parker et al., 1987), \( r_w \) is the ratio of upper-interface resistance to bed resistance with \( r_w = 0.43 \) (Parker et al., 1986), \( \omega_w \) is a fluid entrainment coefficient which is estimated by Parker et al. (1986):

\[ \omega_w = \frac{0.00153}{0.2004 + Ri}, \]

where \( Ri \) is Richardson number defined by:

\[ Ri = \frac{Rg hc}{\sqrt{u^2 + v^2}}. \]

Furthermore, in (1)–(5), \( E = \omega E_0 \) is the sediment entrainment, \( D = \omega \omega_0 \) is the sediment deposition, \( \omega \) is the settling velocity of sediment calculated as (Zhang and Xie, 1993)

\[ \omega = \sqrt{(13.95 v/d)^2 + 1.09(\rho_s/\rho_w - 1)gd - 13.95 v/d}, \]
in which \( \nu \) is the kinematic viscosity of water and \( d \) is the mean diameter of sediment particles, \( r_b = r_b c \) is the local near-bed sediment concentration, and \( r_b \) is a coefficient greater than 1. \( E_i \) is the near-bed concentration at capacity condition and can be estimated by the empirical formula proposed by Parker et al. (1987) which is modified by Hu et al. (2012b):

\[
E_i = \psi_p \frac{1.3 \times 10^{-3} k_m^3}{1 + 4.3 \times 10^{-3} k_m^3}
\]

where \( k_m = \sqrt{C_D R_{ep} |V| / \alpha} \) and \( R_{ep} = \sqrt{R_d \zeta / \nu} \) is the particle Reynolds number and \( \psi_p \) is a correction coefficient.

3. Numerical scheme

Following the coupling approach developed by Fagherazzi and Sun (2003), see also Li and Duffy (2011), the 2-D bed evolution Eq. (5) is rewritten as:

\[
(\beta)_{t} + (hu)c_x + (h v)c_y = 0.
\]

with introducing a new variable \( \beta = (1 - p)Z + hc \), which is the sediment volume in a vertical column above a reference level that can be separated in the sediment volume deposited on the bed \( (1 - p)Z \); and the sediment volume suspended in the fluid.

In order to design a well-balanced numerical scheme for the governing system, the water surface level denoted by \( \eta := h + Z \) is introduced as a primary variable instead of \( h \) in the current study. The “well-balanced” property of the current model is defined in Section 3.4. A coupling strategy is implemented for solving the turbidity currents over erodible bed in the present study; the coupled system (1)–(10) can be rewritten in the following vector form:

\[
U_t + F_x + G_y = S + K.
\]

where the variables \( U \), and the fluxes \( F \) and \( G \) are:

\[
U = \begin{pmatrix}
\eta \\
hu \\
hc \\
\beta
\end{pmatrix}, \\
F = \begin{pmatrix}
hu/n - Z + Z(hc)
\end{pmatrix}, \\
G = \begin{pmatrix}
(hv)/(\eta - Z)
\end{pmatrix},
\]

and the source terms \( S \) and \( K \) are:

\[
S = \begin{pmatrix}
0 \\
-gR(hc)Z_x \\
-gR(hc)Z_y \\
0
\end{pmatrix},
\]

\[
K = \begin{pmatrix}
e_u \sqrt{u^2 + v^2} \\
-(1 + r_{uc})C_D u \sqrt{u^2 + v^2} - \frac{w(u - D)(w - p)}{\rho(1 - p)} - \frac{\rho - \rho_c u}{\rho}\sqrt{u^2 + v^2} \\
-(1 + r_{uc})C_D v \sqrt{u^2 + v^2} - \frac{w(u - D)(w - p)}{\rho(1 - p)} - \frac{\rho - \rho_c v}{\rho}\sqrt{u^2 + v^2} \\
E - D \\
0
\end{pmatrix}.
\]

In this study, the computational domain is discretized by an unstructured triangulation \( T := \bigcup T_i \), consisting of triangular cells \( T_i \) of size \( |T_i| \). The outer unit normal to the corresponding sides of \( T_i \) of length \( \ell_{ik} \), \( k = 1, 2, 3 \), is denoted by \( n_{ik} := (\cos \theta_{ik}, \sin \theta_{ik}) \). The coordinates of the center are denoted by \( (x_i, y_i) \), and the midpoint of the \( k \)th side of the triangle \( T_i \), \( k = 1, 2, 3 \) is denoted by \( M_{ik} = (x_{ik}, y_{ik}) \). The vertices of \( T_i \) are denoted by \( V_{ik} = (x_{ik}, y_{ik}) \). \( k = 12, 23, 31, T_{i1}, T_{i2} \) and \( T_{i3} \) are the neighboring triangles that share a common side with \( T_i \), see Fig. 1.

3.1. Evaluation of numerical fluxes using the central-upwind scheme

In order to solve the system (11), the flux terms: \( F_x + G_y \), have to be computed by a stable and accurate solver, for which the central-upwind scheme is chosen in this study. The central-upwind schemes, which are Riemann problem free shock capturing schemes, enjoy the main advantages of the Godunov-type central schemes, i.e., simplicity, universality and robustness and can be applied to problems with complicated geometries. The readers are referred to Kurganov and Tadmor (2000), Kurganov et al. (2001), Kurganov and Petrova (2005), Kurganov and Petrova (2007), Kurganov and Levy (2002), Bryson et al. (2011) and Liu et al. (2015b, 2015c, 2015a, 2017), where the central-upwind schemes are developed and extended.

Integrating both sides of system (11) over the control volume \( T_i \) and applying the Green’s formula:

\[
\int_{\partial T_i} \nabla \cdot \mathbf{g} \, d \mathit{xy} = \int_{\partial T_i} \mathbf{g} \cdot \mathbf{n} \, d \mathit{s},
\]

one can get a semi-discrete scheme which reads:

\[
\frac{d U_i}{d t} = -\frac{1}{|T_i|} \sum_{k=1}^{3} \ell_{ik} \mathbf{H}_{ik} + \mathbf{S}_i + \mathbf{K}_i.
\]
where $\mathbf{H}_k$ is the normal flux through the side $i_k$. Using the central-upwind scheme, $\mathbf{H}_k$ is estimated by:

$$
\mathbf{H}_k = \cos(\theta_k) \alpha^{in}_{rk} \mathbf{F}(\mathbf{U}(M_k)) + \sin(\theta_k) \alpha^{out}_{rk} \mathbf{F}(\mathbf{U}(M_k))
$$

$$
+ \frac{\alpha_{rk}}{\alpha^{in}_{rk} + \alpha^{out}_{rk}} \left[ \mathbf{U}(M_k) - \mathbf{U}_i(M_k) \right].
$$

(17)

where $\mathbf{U}_i(M_k)$ and $\mathbf{U}_k(M_k)$ are the reconstructed values at $M_k$ using a piecewise linear reconstruction:

$$
\mathbf{U}(x,y) := \mathbf{U}_i + (\mathbf{U}_k)(x-x_i) + (\mathbf{U}_j)(y-y_j), \quad (x,y) \in T_i
$$

(18)

of $\mathbf{U}$, that is:

$$
\mathbf{U}_i(M_k) := \lim_{(x,y) \to M_{(i,y)}(x,y) \in T_i} \mathbf{U}(x,y).
$$

$$
\mathbf{U}_k(M_k) := \lim_{(x,y) \to M_{(k,y)}(x,y) \in T_k} \mathbf{U}(x,y).
$$

(19)

in which the limited gradients $(\mathbf{U}_i)$, $\mathbf{U}_k(M_k)$ and $(\mathbf{U}_j)$ are component-wise approximations of the gradients $\mathbf{U}_i(x_i, y_i, t)$ and $\mathbf{U}_j(x_j, y_j, t)$, respectively, computed using a nonlinear slope limiter to minimize the oscillations of reconstruction (18), see Section 3.2 for the computation of $\mathbf{U}_i$, $\mathbf{U}_k$, $(\mathbf{U}_i)$, and $(\mathbf{U}_j)$. In order to maintain numerical stability when reconstruction (18) is applied, the appearance of local extrema is checked. If the condition:

$$
\min(\mathbf{U}_i, \mathbf{U}_k) \leq \mathbf{U}_i(M_k) \leq \max(\mathbf{U}_i, \mathbf{U}_k), \quad k = 1, 2, 3
$$

(20)

is not satisfied for cell $T_i$ ($\mathbf{U}_k(M_k) = (\mathbf{U}_k) = 0$ is used in the reconstruction (18). Furthermore, $Z$ is also reconstructed over $T_i$ using (18), and $\beta$ is reconstructed by:

$$
\beta(x,y) := (1-p)\hat{Z}(x,y) + hc(x,y), \quad (x,y) \in T_i.
$$

(21)

For the dry cells in which $\hat{T}_i = Z_i$, the reconstructions of $\eta$, $hu$, $hv$ and $hc$ are recalculated as:

$$
\hat{\eta}(x,y) = \hat{Z}(x,y), \quad hu(x,y) = 0, \quad hv(x,y) = 0, \quad hc(x,y) = 0, \quad (x,y) \in T_i
$$

(22)

where $\hat{Z}(x,y)$ is computed by the piecewise linear reconstruction (18), and (21) is recomputed based on (22).

Cell averages of the source terms $S$ and $K$ in (16) need to be discretized in an appropriate form. In this study, special discretization of the source terms are proposed in order to guarantee the well-balanced property of the proposed scheme, as described in Section 3.4.

In (17), $\alpha^{in}_{rk}$ and $\alpha^{out}_{rk}$ are one-sided local speeds. In order to avoid division by zero, the definitions of $\alpha^{in}_{rk}$ and $\alpha^{out}_{rk}$ in Bryson et al. (2011) are modified in this study. They are defined by:

$$
\alpha^{in}_{rk} = -\min(\lambda_1 J_k(\mathbf{U}(M_k)), \lambda_1 J_k(\mathbf{U}(M_k)), -\beta),
$$

$$
\alpha^{out}_{rk} = \max(\lambda_5 J_k(\mathbf{U}(M_k)), \lambda_5 J_k(\mathbf{U}(M_k)), \beta),
$$

(23)

in which $\beta = 10^{-8}$ is a small positive number, $\lambda_1 J_k(\mathbf{U}(M_k)) \leq \cdots \leq \lambda_5 J_k(\mathbf{U}(M_k))$ are the eigenvalues of the Jacobian matrix:

$$
J_k = \cos(\theta_k) \frac{\partial \mathbf{F}}{\partial \mathbf{U}} + \sin(\theta_k) \frac{\partial \mathbf{G}}{\partial \mathbf{U}}.
$$

(24)

where

$$
\frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
-\frac{(hu)^2}{(\eta-Z)^2} & \frac{2(hu)}{\eta-Z} & 0 & \frac{R}{\eta-Z} & 0 \\
-\frac{(hu)(hv)}{(\eta-Z)^2} & \frac{hv}{\eta-Z} & 0 & 0 & 0 \\
-\frac{(hu)}{(\eta-Z)^2} & \frac{hc}{\eta-Z} & 0 & 0 & 0 \\
-\frac{(hu)}{(\eta-Z)^2} & \frac{hc}{\eta-Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\frac{\partial \mathbf{G}}{\partial \mathbf{U}} = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
-\frac{(hv)(hc)}{(\eta-Z)^2} & \frac{hv}{\eta-Z} & \frac{2(hv)}{\eta-Z} & \frac{R}{\eta-Z} & 0 \\
-\frac{(hv)(hc)}{(\eta-Z)^2} & \frac{hv}{\eta-Z} & 0 & 0 & 0 \\
-\frac{(hv)(hc)}{(\eta-Z)^2} & \frac{hv}{\eta-Z} & 0 & 0 & 0 \\
-\frac{(hv)(hc)}{(\eta-Z)^2} & \frac{hv}{\eta-Z} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

(25)

The analytical expressions of the eigenvalues of (24) are:

$$
\lambda_1 = \frac{(hu) \cos(\theta_k) + (hv) \sin(\theta_k)}{\eta-Z} - \sqrt{gR(hc)},
$$

$$
\lambda_2 = \lambda_3 = \frac{(hu) \cos(\theta_k) + (hv) \sin(\theta_k)}{\eta-Z},
$$

$$
\lambda_4 = 0, \quad \lambda_5 = \frac{(hu) \cos(\theta_k) + (hv) \sin(\theta_k)}{\eta-Z} + \sqrt{gR(hc)}.
$$

(26)

Moreover, in order to correctly predict “lake at rest” steady states (see the descriptions in Section 3.4), the fourth component of (17) is computed using a centered scheme at interface $i_k$ when $\eta_i(M_k) = \eta_k(M_k)$, $c_i = c_k$, and $u_i(M_k) = u_k(M_k)$, $v_i(M_k) = v_k(M_k) = 0$, where the velocities $u$ and $v$ at $M_k$ are given by (33).

A fully discrete scheme can be obtained from the semidiscretization (16) by integrating the time-dependent terms using a stable and sufficiently accurate ODE solver. In the current study, the third-order strong stability preserving (SSP) Runge–Kutta solver, see, e.g., Gottlieb et al. (2011, 2001), is used. To maintain the numerical stability, the time step size should satisfy the CFL condition, see Kurganov and Petrova (2005), which can be expressed as:

$$
\Delta t < \frac{1}{3} \min_{i,k} \left[ \frac{r_{ik}}{\max(\alpha^{in}_{rk}, \alpha^{out}_{rk})} \right],
$$

(27)

where $r_{ik}$, $k = 1, 2, 3$ are the three corresponding altitudes of triangle $T_i \in T$.

3.2. Estimations of the gradients

The gradients $(\mathbf{U}^T \hat{\mathbf{V}})^T = [(\mathbf{U}_i)^T, (\mathbf{U}_j)^T]$ of cell $T$ are estimated by taking the $x$- and $y$-derivatives of the planes passing through the points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$, $(x_4, y_4)$, $(x_5, y_5)$ (the gray area in Fig. 2):

$$
(\mathbf{U}_i)^T = \frac{(y_3 - y_1)(\mathbf{U}_2 - \mathbf{U}_1) - (y_2 - y_1)(\mathbf{U}_3 - \mathbf{U}_1)}{(y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)}
$$

(28)

$$
(\mathbf{U}_j)^T = \frac{(x_5 - x_3)(\mathbf{U}_2 - \mathbf{U}_1) - (x_2 - x_1)(\mathbf{U}_3 - \mathbf{U}_1)}{(x_5 - x_3)(y_2 - y_1) - (x_2 - x_1)(y_3 - y_1)}. \quad \text{(28)}
$$

The use of the gradients estimated by (28) in reconstruction (18) can lead to oscillations and instabilities. In order to minimize the numerical oscillations, the gradients used in the piecewise linear reconstruction (18) have to be suppressed by a slope limiter.

In the present study, the multidimensional slope limiter developed by Jawahar and Kamath (2000) is used to obtain the limited
gradients $(\hat{U}_h)$ and $(\hat{U}_h)$:

\[
\begin{align*}
(\hat{U}_h)_i &= \Lambda_{i1}(U_h)_1 + \Lambda_{i2}(U_h)_2 + \Lambda_{i3}(U_h)_3, \\
(\hat{U}_h)_i &= \Lambda_{i1}(U_h)_1 + \Lambda_{i2}(U_h)_2 + \Lambda_{i3}(U_h)_3,
\end{align*}
\]

where $\Lambda_{i1}$, $\Lambda_{i2}$ and $\Lambda_{i3}$ are weights which are defined by:

\[
\begin{align*}
\Lambda_{i1} &= \frac{\| (\nabla U)_{1i} \|^2 + \| (\nabla U)_{2i} \|^2 + \| (\nabla U)_{3i} \|^2 + 3\varepsilon}{h^2}, \\
\Lambda_{i2} &= \frac{\| (\nabla U)_{1i} \|^2 + \| (\nabla U)_{2i} \|^2 + \| (\nabla U)_{3i} \|^2 + 3\varepsilon}{h^2}, \\
\Lambda_{i3} &= \frac{\| (\nabla U)_{1i} \|^2 + \| (\nabla U)_{2i} \|^2 + \| (\nabla U)_{3i} \|^2 + 3\varepsilon}{h^2},
\end{align*}
\]

where the parameter $\varepsilon$ is a small positive number introduced to prevent division by zero. In this paper, $\varepsilon = 10^{-14}$ is used.

### 3.3. Nonnegative piecewise linear reconstruction for $Z$

Notice that the use of the reconstruction (18) for $\eta$ and $Z$ cannot guarantee that the reconstructed values of the water surface level $\eta_i(M_{ik})$ are above the reconstructed bed level $Z_i(M_{ik})$. This introduces unphysical negative water depths at $M_{ik}$. Therefore, the correction of the piecewise linear reconstruction for $Z$ or $\eta$ is necessary to ensure that $\eta_i(M_{ik}) := \eta_i(M_{ik}) - Z_i(M_{ik}) > 0$. In Bryson et al. (2011), a 2-D correction algorithm is applied to $\eta$. However, such correction of $\eta$ may fail in maintaining the “lake at rest” steady states at wetting and drying fronts for the studied system. Thus, the correction algorithm from Bryson et al. (2011) is modified in the present paper to reconstruct $Z$. Notice that such corrections should be taken only for those problematic cells.

For the wet cells in which $Z_i < \eta_i$, it is impossible to have $\hat{Z}^{\text{new}}(x_{ik}, y_{ik}) \geq \eta_i(x_{ik}, y_{ik})$ for all three vertices of cell $T_i$. Thus, only two problematic cases should be considered:

- **Case 1**: There is only one vertex, for example, with index $\kappa = 12$, for which $\hat{Z}_i(x_{ik}, y_{ik}) > \eta_i(x_{ik}, y_{ik})$.

  In this case, $\hat{Z}_i(x_{ik}, y_{ik}) = \eta_i(x_{ik}, y_{ik})$ is first set for the problematic vertex with $\kappa = 12$. Then, based on the conservation requirement, $\hat{Z}_i(x_{ik}, y_{ik}) - \frac{1}{2}(\eta_i(x_{ik}, y_{ik}) - Z_i)$ is set for other two vertices with $\kappa = 23, 31$. The original piecewise linear reconstruction (18) for $Z$ is replaced with a new one defined by:

\[
\begin{align*}
\begin{vmatrix}
 x - x_1 & y - y_1 & \hat{Z}(x, y) - \hat{Z}_i \\
 x_{12} - x_1 & y_{12} - y_1 & \frac{1}{2}(\eta_i(x_{12}, y_{12}) - Z_i) \\
 x_{23} - x_1 & y_{23} - y_1 & \frac{1}{2}(\eta_i(x_{23}, y_{23}) - Z_i) \\
(x, y) \in T_i
\end{vmatrix}
= 0.
\end{align*}
\]

- **Case 2**: There are two vertices, for example, with indexes $\kappa = 12, 23$, for which $\hat{Z}_i(x_{ik}, y_{ik}) > \eta_i(x_{ik}, y_{ik})$.

  In this case, $\hat{Z}_i(x_{ik}, y_{ik}) = \eta_i(x_{ik}, y_{ik})$ is first set for these two problematic vertices with $\kappa = 12, 23$. Then, based on the conservation requirement, $\hat{Z}_i(x_{ik}, y_{ik}) - \frac{3}{2}(\eta_i(x_{ik}, y_{ik}) - Z_i)$ is set for another vertex with $\kappa = 31$. The original piecewise linear reconstruction (18) for $Z$ is replaced with a new one defined by:

\[
\begin{align*}
\begin{vmatrix}
 x - x_1 & y - y_1 & \hat{Z}(x, y) - \hat{Z}_i \\
 x_{12} - x_1 & y_{12} - y_1 & \eta_i(x_{12}, y_{12}) - Z_i \\
 x_{23} - x_1 & y_{23} - y_1 & \eta_i(x_{23}, y_{23}) - Z_i \\
(x, y) \in T_i
\end{vmatrix}
= 0, \quad (x, y) \in T_i.
\end{align*}
\]

The above described correction yields a new linear reconstruction for $Z$ which is conservative, and guarantees that the corrected $\hat{Z}(x, y)$ is not greater than the reconstructed $\eta_i(x, y)$ over the whole computational domain.

In order to avoid the division by very small water depth $h := \eta - Z$, as proposed in Kurganov and Petrova (2007), the desingularized point values of $u$, $v$ and $c$ are computed by (i, k indexes are omitted):

\[
\begin{align*}
u &= \frac{\sqrt{2} h (hu)}{\sqrt{h^4 + \max(h^4, \delta)}}, \\
v &= \frac{\sqrt{2} h (hv)}{\sqrt{h^4 + \max(h^4, \delta)}}, \\
c &= \frac{\sqrt{2} h (hc)}{\sqrt{h^4 + \max(h^4, \delta)}},
\end{align*}
\]

where $\delta$ is a prescribed tolerance; $\delta = 10^{-16}$ is used in the present study. Notice that the choice of the value of $\delta$ is relevant with the precision of computer number format. In the current study, the computer program for testing the proposed model is double precision.

Using the desingularized point values computed in (33), the $x$- and $y$-discharges and fluxes are recalculated by:

\[
\begin{align*}
F &= \left( (hu), (hu) \cdot u + \frac{g}{2} R(\eta - Z)(hc), (hu) \cdot v, (hc) \cdot u, (hc) \cdot v \right)^T, \\
G &= \left( (hv), (hu) \cdot v, (Dv) \cdot v + \frac{g}{2} R(\eta - Z)(hc), (hc) \cdot v, (hc) \cdot v \right)^T,
\end{align*}
\]

and the one-sided local speeds (23) are computed using the following formulas:

\[
\begin{align*}
a_{\text{max}} &= -\min \{ u_i^+ (M_{ik}) - \sqrt{g R_i (M_{ik})(hc)_{ik} (M_{ik})} \}, \\
u_i^+ (M_{ik}) &= \frac{1}{\sqrt{g R_i (M_{ik})(hc)_{ik} (M_{ik})}} \| (\nabla U)_{ik} \|, \\
a_{\text{max}} &= \max \{ u_i^+ (M_{ik}) + \sqrt{g R_i (M_{ik})(hc)_{ik} (M_{ik})} \}, \\
u_i^+ (M_{ik}) &= \frac{1}{\sqrt{g R_i (M_{ik})(hc)_{ik} (M_{ik})}} \| (\nabla U)_{ik} \|
\end{align*}
\]

where $R_i(M_{ik})$ and $R_k(M_{ik})$ are computed using $c_i(M_{ik})$ and $c_k(M_{ik})$ estimated by (33), respectively; the normal velocities denoted by $u_i^+(M_{ik})$ and $u_k^+(M_{ik})$ at the point $M_{ik}$ are given by:

\[
\begin{align*}
u_i^+(M_{ik}) &= u_i(M_{ik}) \cos(\theta_i) + v_i(M_{ik}) \sin(\theta_i), \\
u_k^+(M_{ik}) &= u_k(M_{ik}) \cos(\theta_k) + v_k(M_{ik}) \sin(\theta_k),
\end{align*}
\]

### 3.4. Discretization of the source terms

A desired property of discretization of bed-slope source term $S_i$ is well-balanced property, which guarantees that the proposed discretization of $S_i$ exactly balances the numerical fluxes under the “lake at rest” steady states. The “lake at rest” steady solutions should satisfy that:

\[
\eta = \eta_0 \equiv \text{Const.}, \quad u = v = 0.
\]
In the “lake at rest” states (37), one can get $a_{ik}^{in} = a_{ik}^{out}$ by (35). After substituting the “lake at rest” states into (16), one can conclude that the well-balanced discretization of $S_i$ should satisfy the following conditions:

$$
\frac{g}{|T_i|} \sum_{k=1}^{3} \epsilon_{ik} \cos(\theta_{ik}) \frac{1}{2} \left[ \frac{R_k(M_{ik})(hc)_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})}{2} \right] \eta_i
$$

$$
- \frac{R_k(M_{ik})(hc)_{ik}(M_{ik})Z_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})Z_i(M_{ik})}{2} \right] \eta_i
$$

$$
+ S^{(2)}_i = 0
$$

(38)

and

$$
\frac{g}{|T_i|} \sum_{k=1}^{3} \epsilon_{ik} \sin(\theta_{ik}) \frac{1}{2} \left[ \frac{R_k(M_{ik})(hc)_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})}{2} \right] \eta_i
$$

$$
- \frac{R_k(M_{ik})(hc)_{ik}(M_{ik})Z_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})Z_i(M_{ik})}{2} \right] \eta_i
$$

$$
+ S^{(3)}_i = 0
$$

(39)

in which

$$
S^{(2)}_i \approx - \frac{g}{|T_i|} \int_{T_i} \text{RhcZ}_{ik} \, dxdy,
$$

$$
S^{(3)}_i \approx - \frac{g}{|T_i|} \int_{T_i} \text{RhcZ}_{ik} \, dxdy.
$$

(40)

Following the procedures given in Bryson et al. (2011), the Green's formula (15) is first applied to the vector field $\mathbf{g} = (\frac{1}{2}Rh^2, 0)$, and one can obtain:

$$
\frac{g}{|T_i|} \sum_{k=1}^{3} \int_{\partial T_i} \left( Rhc \cos(\theta_{ik}) \right) ds
$$

$$
= \int_{T_i} \text{RhcZ}_{ik} \, dxdy + \int_{T_i} \frac{1}{2} \text{RhcZ}_{ik} \, dxdy + \int_{T_i} \frac{1}{2} \text{RhcZ}_{ik} \, dxdy.
$$

(41)

Based on the fact that $h = \eta - Z$ and $R_k = \eta_k - Z_k$, (41) can be further reformulated as:

$$
- \int_{T_i} \text{RhcZ}_{ik} \, dxdy = \frac{g}{|T_i|} \sum_{k=1}^{3} \int_{\partial T_i} \left( Rhc \eta - \text{RhcZ} \right) \cos(\theta_{ik}) ds
$$

$$
- \int_{T_i} \text{RhcZ}_{ik} \, dxdy - \int_{T_i} \frac{1}{2} \text{RhcZ}_{ik} \, dxdy
$$

$$
- \int_{T_i} \frac{1}{2} \left( \eta - Z \right) \text{RhcZ}_{ik} \, dxdy.
$$

(42)

where $(\partial T_i)_k$ is the kth side of the triangle $T_i$, $k = 1, 2, 3$. Next, one can apply the midpoint rule to the integrals on the RHS of (42) and obtain the following discretization of $S^{(2)}_i$:

$$
S^{(2)}_i = \frac{g}{|T_i|} \sum_{k=1}^{3} \epsilon_{ik} \cos(\theta_{ik}) \frac{1}{2} \left[ \frac{R_k(M_{ik})(hc)_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})}{2} \right] \eta_i(M_{ik})
$$

$$
\frac{R_k(M_{ik})(hc)_{ik}(M_{ik})Z_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})Z_i(M_{ik})}{2} \right] \eta_i(M_{ik})
$$

$$
- gR(hc)_{ik} \eta_i - gR \left( \frac{1}{2} (\eta_i - Z) \right)^2 (c_y) - g \left( \frac{1}{2} (\eta_i - Z) \right) (\eta_i).
$$

(43)

Similarly, one can obtain the discretization of $S^{(3)}_i$:

$$
S^{(3)}_i = \frac{g}{|T_i|} \sum_{k=1}^{3} \epsilon_{ik} \sin(\theta_{ik}) \frac{1}{2} \left[ \frac{R_k(M_{ik})(hc)_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})}{2} \right] \eta_i(M_{ik})
$$

$$
\frac{R_k(M_{ik})(hc)_{ik}(M_{ik})Z_{ik}(M_{ik}) + R_i(M_{ik})(hc)(M_{ik})Z_i(M_{ik})}{2} \right] \eta_i(M_{ik})
$$

$$
- gR(hc)_{ik} \eta_i - gR \left( \frac{1}{2} (\eta_i - Z) \right)^2 (c_y) - g \left( \frac{1}{2} (\eta_i - Z) \right) (\eta_i).
$$

(44)

The gradients $\eta$, $\eta_y$, $c_x$, $c_y$, $R_k$ and $R_i$ are calculated using (28). Since $\eta = \eta_y = 0$ for $\eta = \text{constant}$, $c_x = c_y = 0$ and $R_k = R_i = 0$ for $c = \text{constant}$ in the “lake at rest” states, the quadratures (43)–(44) satisfy (38)–(39), respectively.

The source term $\mathbf{K}_i$ is discretized straightforwardly using the midpoint rule:

$$
\mathbf{K}_i = \mathbf{K}(x_i, y_i),
$$

and it does not affect the well-balanced property of the developed numerical scheme.

Notice that, for the turbidity flows driven by suspended sediment over irregular beds, the “lake at rest” steady status given by (37) cannot be preserved in the real-world applications due to the fact that sediment deposition will cause the density differences and horizontal advection. Such steady status is only an ideal condition defined to facilitate the developments and tests of “well-balanced” property. In order to preserve such steady status, one have to force $E = D = 0$, see test 4.1.

3.5. Positivity preserving property of the proposed scheme

Based on the fact that the maximum amount of deposition in one time step should not be greater than the amount of suspended sediment in the fluid, in the present study, the following condition is specified:

$$
\Delta t \leq \sigma (\text{Rhc}),
$$

(45)

where $\sigma$ is a coefficient specified in the range $0 < \sigma < 1$, and $\sigma = 0.5$ is used in this study. Based on the empirical relation $D = \text{ortg} T_i$ given in Section 2, using that $(\text{Rhc})_i = \text{Rhc}_i$, (45) can be reformulated as:

$$
\Delta t \leq \min_i \left( \frac{1}{2} \frac{\text{Rhc}_i}{\text{ortg}_i} \right).
$$

(46)

It can be noted that (46) may lead to very small $\Delta t$ when water depth $\text{Rhc}_i$ is close to zero in any one cell of the whole computational domain. To avoid such inefficiency in practical applications, a threshold $\text{Rhc}_i = 1 \times 10^{-6}$ is set in the current study, which implies that (46) is only used for those cells in which the water depth is no less than the specified threshold. Notice that this specified threshold is only used for (46) to ensure the positivity preserving property; it does not affect dealing with wetting and drying by proposed model.

Using the forward Euler temporal discretization, and subtracting the fourth and fifth components from the first component of (16), one can obtain:

$$
\text{Rhc}_{i+1} = \text{Rhc}_i - \Delta t \frac{3}{|T_i|} \sum_{k=1}^{3} \epsilon_{ik} \cos(\theta_{ik}) \left[ a_{ik}^{in} (hu)_{ik}(M_{ik}) + a_{ik}^{out} (hu)_{ik}(M_{ik}) \right]
$$

$$
- \Delta t \frac{3}{|T_i|} \sum_{k=1}^{3} \epsilon_{ik} \sin(\theta_{ik}) \left[ a_{ik}^{in} (hv)_{ik}(M_{ik}) + a_{ik}^{out} (hv)_{ik}(M_{ik}) \right]
$$

$$
+ \Delta t \frac{3}{|T_i|} \sum_{k=1}^{3} \epsilon_{ik} \cos(\theta_{ik}) \left[ a_{ik}^{in} (hu)_{ik}(M_{ik}) + a_{ik}^{out} (hu)_{ik}(M_{ik}) \right]
$$

$$
+ \Delta t \epsilon \sqrt{u^2 + v^2} + \Delta t \frac{E - D}{1 - p}.
$$

(47)

where the subscripts $n$ and $n+1$ represent the time $t = t^n$ and $t = t^{n+1}$, respectively.
Based on (34) and (36), and the fact that \( \mathcal{H}_i^n = \frac{1}{2} \sum_{k=1}^{3} h_i(M_{jk}) \), (47) can be reformulated as:
\[
\mathcal{H}_i^{n+1} = \Delta t \left[ \sum_{k=1}^{3} h_i(M_{jk}) \frac{\ell_k a_{out}^{in} + a_{out}^{out}}{a_{in}^{in} + a_{out}^{out}} \left[ a_{in}^{in} - u_{ik}^+(M_{jk}) \right] \right] \\
+ \Delta t e \sqrt{u^2 + v^2 + \Delta t \left[ F \right]} \\
+ \Delta t \left[ \sum_{k=1}^{3} h_i(M_{jk}) \left[ 1 \right] \frac{\Delta t}{|T|} \frac{\ell_k a_{out}^{in} + a_{out}^{out}}{a_{in}^{in} + a_{out}^{out}} \left[ a_{in}^{in} + u_{ik}^+(M_{jk}) \right] \right] - \Delta t D. \\
\]
(48)

One can get \( a_{out}^{out} \geq u_{ik}^+(M_{jk}) \) from (35) and therefore, the first three terms on the RHS of (48) are nonnegative. Using \( \mathcal{H}_i^n = \frac{1}{2} \sum_{k=1}^{3} h_i(M_{jk}) \) and (45), one can obtain the following condition to guarantee that \( \mathcal{H}_i^{n+1} \) is nonnegative:
\[
\sum_{k=1}^{3} h_i(M_{jk}) \left[ 1 \right] \frac{\Delta t}{|T|} \frac{\ell_k a_{out}^{in} + a_{out}^{out}}{a_{in}^{in} + a_{out}^{out}} \left[ a_{in}^{in} + u_{ik}^+(M_{jk}) \right] - \Delta t D \\
\geq 0. \\
\] (49)

Using \( u_{ik}^+(M_{jk}) \leq a_{out}^{out} \) from (35), condition (49) gives:
\[
\frac{\Delta t}{|T|} \frac{\ell_k a_{out}^{out}}{a_{in}^{in} + a_{out}^{out}} \left[ a_{in}^{in} + u_{ik}^+(M_{jk}) \right] + \frac{\gamma_i}{6} \frac{\ell_k}{1 - p} \\
\leq \frac{\Delta t}{|T|} \frac{\ell_k a_{out}^{out}}{1 - p} \frac{\gamma_i}{6} \frac{\ell_k}{1 - p} \leq \frac{1}{3}. \\
\] (50)

Based on (50) and the fact that \( |T| = 0.5 r_k l_k \), the following limitation has to be satisfied to ensure that \( \mathcal{H}_i^{n+1} \) computed by (47) is nonnegative, that is:
\[
\Delta t \leq \frac{1}{6} \min_{k} \left[ \frac{r_k}{\max(a_{in}^{in} \ell_k a_{out}^{out})} \right] \left( 1 - \frac{0.5 \gamma_i}{1 - p} \right). \\
\] (51)

Another desired property of the proposed numerical model is that the positivity preserving property for sediment concentration \( c_i^n \). Based on the fact that \( \mathcal{H}_i^{n+1} \geq 0 \) guaranteed by (51), in order to prove that \( \mathcal{H}_i^{n+1} \geq 0 \), one only needs to ensure that \( \mathcal{H}_i^{n+1} \geq 0 \). Using the forward Euler temporal discretization, the fourth component in Eq. (16) can be written in a fully discrete form:
\[
\mathcal{H}_i^{n+1} = \mathcal{H}_i^n + \Delta t \sum_{k=1}^{3} \ell_k \cos(\theta_k) \left[ a_{in}^{in}(hu)_{jk}(M_{jk}) c_k(M_{jk}) + a_{out}^{out}(ht)_{jk}(M_{jk}) c_j(M_{jk}) \right] \\
- \Delta t \sum_{k=1}^{3} \ell_k \sin(\theta_k) \left[ a_{in}^{in}(hu)_{jk}(M_{jk}) c_k(M_{jk}) + a_{out}^{out}(ht)_{jk}(M_{jk}) c_j(M_{jk}) \right] \\
+ \Delta t \sum_{k=1}^{3} \ell_k \cos(\theta_k) \left[ a_{in}^{in}(hu)_{jk}(M_{jk}) c_k(M_{jk}) - a_{out}^{out}(ht)_{jk}(M_{jk}) c_j(M_{jk}) \right] \\
+ \Delta t (E - D). \\
\] (52)

Based on (34) and (36), and the fact that \( \mathcal{H}_i^{n+1} = \frac{1}{2} \sum_{k=1}^{3} (h_i c_k)(M_{jk}) \), (52) can be reformulated as:
\[
\mathcal{H}_i^{n+1} = \Delta t \sum_{k=1}^{3} \ell_k h_i(M_{jk}) c_k(M_{jk}) \left[ a_{in}^{in} - u_{ik}^+(M_{jk}) \right] \\
+ \Delta t E \sum_{k=1}^{3} h_i(M_{jk}) c_j(M_{jk}) \\
\left[ 1 + \frac{6}{\ell_k a_{out}^{out}} \left( a_{in}^{in} + u_{ik}^+(M_{jk}) \right) \right] - \Delta t D. \\
\] (53)

One can get \( a_{out}^{out} \geq u_{ik}^+(M_{jk}) \) from (35) and therefore, the first two terms on the RHS of (53) are nonnegative. Accordingly, the sum of the last two terms on the RHS of (53) needs to be nonnegative to guarantee that \( \mathcal{H}_i^{n+1} \geq 0 \). Using (34) and (45), one can get \( \mathcal{H}_i^n = \frac{1}{3} \sum_{k=1}^{3} h_i(M_{jk}) c_k(M_{jk}) \), and this gives the following condition:
\[
\sum_{k=1}^{3} h_i(M_{jk}) c_k(M_{jk}) \left[ 1 \right] \frac{\Delta t}{|T|} \frac{\ell_k a_{out}^{out}}{a_{in}^{in} + a_{out}^{out}} \left[ a_{in}^{in} + u_{ik}^+(M_{jk}) \right] - \frac{1}{6} \frac{\ell_k}{1 - p} \\
\geq 0. \\
\] (54)

Using \( u_{ik}^+(M_{jk}) \leq a_{out}^{out} \) from (35), condition (54) gives:
\[
\Delta t \frac{\ell_k a_{out}^{out}}{a_{in}^{in} + a_{out}^{out}} \left[ a_{in}^{in} + u_{ik}^+(M_{jk}) \right] \leq \frac{\Delta t}{|T|} \frac{\ell_k a_{out}^{out}}{1 - p} \leq 1. \\
\] (55)

Based on (55) and the fact that \( |T| = 0.5 r_k l_k \), the following limitation has to be satisfied to ensure that \( \mathcal{H}_i^{n+1} \) computed by (52) is nonnegative, that is:
\[
\Delta t \leq \frac{1}{6} \min_{k} \left[ \frac{r_k}{\max(a_{in}^{in} \ell_k a_{out}^{out})} \right]. \\
\] (56)

Based on the fact that \( 0.5 \leq (1 - 0.5 \gamma_i)/p < 1 \), conditions (27), (51) and (56) can be combined into one single limitation which is (56). Notice that (56) is a stricter CFL condition than (27) and can ensure the stability of the numerical method.

Finally, to guarantee the positivity preserving property of the proposed numerical scheme while maintaining the numerical stability, \( \Delta t \) should satisfy the restrictions (46) and (56) at the same time for each time step.

3.6. “Lake at rest” steady solutions at wetting and drying fronts

The nonnegative piecewise linear reconstruction and desingularization described in Section 3.3 and positivity preserving property described in Section 3.5 can successfully deal with the wetting and drying, and it does not require a minimum fluid depth and concentration to be set. The “well-balanced” discretization of bed slope term described in Section 3.4 can successfully guarantee that the “Lake at rest” steady solutions (37) are satisfied for wet domain. In order to further guarantee the “lake at rest” steady states at wetting-drying fronts, the following treatments are used.

When calculating the numerical fluxes using (17) for interfaces \( \ell_k \) between a quiescent wet cell and a dry cell, the gradients of variables at such interfaces are temporally zeroed when \( \tilde{\gamma}_i \) of the wet cell is no higher than the \( \tilde{\gamma}_i \) of the neighboring dry cell, using a zero-order extrapolation condition. This treatment is important to eliminate the unphysical waves and oscillations at wetting-drying fronts under the “lake at rest” status (37).

Moreover, using the weighted slope limiter (29) cannot guarantee that the “lake at rest” solution (37) is satisfied at wetting and drying fronts due to the fact that the gradient \( \eta_k \) and \( \eta_k \) of the neighboring dry cell is not zero. To avoid this, \( \tilde{\eta}_k = (\tilde{\eta}_k) = 0 \) is used in (18) for the quiescent wet cells which have at least one dry neighboring cell.

4. Numerical experiments

In this section, we demonstrate the performance of the developed numerical model on four test problems.
4.1. “Lake at rest” steady status over irregular beds with wet-dry interfaces

This test case is designed to demonstrate the well-balanced property of the proposed scheme, which is guaranteed by the proposed discretization of the bed slope source terms (43)-(44). The well-balanced property is evaluated not only in the wet domain but also at the boundaries between wet and dry zones. To this end, two islands with different altitudes are defined as

\[
Z(x, y) = \max \{0 \text{ m}, 1 \text{ m} - 0.8 \cdot [(x - 4 \text{ m})^2 + (y - 2 \text{ m})^2], \\
0.5 - 1.2 \cdot [(x - 2 \text{ m})^2 + (y - 2 \text{ m})^2]\} 
\]

(57)

in a [0 m, 2 m] × [0 m, 1 m] computational domain, and sediment entrainment and deposition are not considered in this test (Figs. 3–5).

A quiescent lake around the two islands is defined with an initial water surface elevation of 0.6 m, so that the test can involve the wet-dry interfaces for the big island with a height of 1 m. The initial sediment concentration in the water is set to 0.01, the mean diameter of the sediment is set to 0.1 mm and the porosity of the saturated bed is 0.45. The domain is divided into 9600 triangular cells. Fig. 8 shows the 3-D views of the simulated still water surface at t = 50 s using the proposed well-balanced model.
An undisturbed water surface is observed throughout the entire simulation.

In order to show the necessity and importance of the proposed well-balanced discretization \((43)-(44)\), a non-well-balanced version of the current proposed model is introduced and applied to the same numerical test for comparison purposes. The non-well-balanced version of the proposed model is obtained by replacing the bed-slope terms \(S^{(2)}_i\) and \(S^{(3)}_i\) in \((38)\) and \((39)\) by a straightforward midpoint rule discretization:

\[
\begin{align*}
S^{(2)}_i &= -R(x_i, y_i) \sum_{k=1}^{3} \sum_{h=1}^{M} \frac{g}{|T_i|} \left( \frac{L}{M} \right) \cos(\theta_{ik}) \left( Z_i(M_h) + \frac{Z_h(M_h)}{2} \right), \\
S^{(3)}_i &= -R(x_i, y_i) \sum_{k=1}^{3} \sum_{h=1}^{M} \frac{g}{|T_i|} \left( \frac{L}{M} \right) \sin(\theta_{ik}) \left( Z_i(M_h) + \frac{Z_h(M_h)}{2} \right).
\end{align*}
\]

Using the same grid, the test runs 50 s for the well-balanced model however can only run 1.03 s for the non-well-balanced model due to the large oscillations generated at the wet-dry boundaries. Figs. 9 and 10 show the simulated discharges at \(y = 2 \) m cross sections and simulated volumetric concentrations at \(x = 2 \) m cross sections using the well-balanced and the non-well-balanced schemes, respectively.

It can be clearly observed that spurious waves are generated by the non-well-balanced scheme at the plateau of the submerged island and the wet-dry boundaries as shown in Fig. 9 (b), which leads to a wrong prediction of sediment concentration in these areas as illustrated in Fig. 10 (b). This clearly demonstrates the importance and necessity of the well-balanced property of the current model guaranteed by the proposed scheme \((43)-(44)\).

### 4.2. Sediment-driven turbidity current in a channel over initially dry beds

This experimental test is used to verify the accuracy of the proposed numerical model. The laboratory tests, modeling the turbidity currents over submarine canyons and the formed deposition fans, were reported in García (1993). The experiments were conducted in a 30 cm wide and 11.6 m long channel which consists of a 5 m long inclined bed with a slope of 0.08, followed by a 6.6 m long horizontal bed. The flume is filled by clear water as ambient fluid, and the initial bed is treated as a dry bed with \(h = 0 \) m. A sediment-laden bottom current of known density was released at the top of the inclined bed at a known rate, and the current starts to flow downslope at \(t = 0 \) s. The inlet current thickness \(h_0\) was set to be 3 cm with a certain layer-averaged inlet velocity \(u_0\), see Fig. 6.

Notice that, since the initially dry bed is non-erodible, the updated bed level can not be less than the initial bed level. In the numerical simulations, a 11.6 m \(\times\) 0.4 m computational domain is equally divided into 18,560 triangular cells, and it is found that finer mesh leads to essentially same numerical results. The modeling parameters of several selected experimental runs in García (1993) are given in Table 1.

Fig. 7 shows the comparisons between computed and measured results of deposition patterns against distance. Good agreements can be observed for selected runs. This shows that the proposed model can accurately simulate the sediment-driven turbidity.

### 4.3. 2-D axisymmetric particle-driven turbidity currents

Two-dimensional experiments of axisymmetric particle-driven current by Bonnecaze et al. (1995) is numerically investigated using the proposed model. The experiments were performed to determine the radius of a fixed-volume gravity current at different times and its resulting deposition pattern. A plan view sketch of the laboratory tank is given in Fig. 8. At \(t = 0\) s, the lock gate was suddenly lifted to release the turbid fluid stored in the upstream reservoir, i.e. the rectangular part with gray color in Fig. 8. The downstream side of the lock gate was initially filled with clear water. The initial thickness of the well-mixed turbidity in the reservoir is 0.14 m. The suspension for the current was made by non-cohesive silicon carbide particles with a diameter of 37 \(\mu\)m, a density of \(\rho_s = 3217 \) kg/m\(^3\), a settling velocity of 0.17 cm/s and a
porosity of 0.5. Three experimental runs with different initial sediment concentrations (0.0193, 0.00966 and 0.00506) were utilized. In the numerical simulations, the computational domain is discretized into 12,174 unstructured triangular cells. The drag coefficient $C_D$ is set to 0.02, $\psi_p$ is set to 0.15 and $r_s$ is set to 1 according to Bradford and Katopodes (1999).

Fig. 9(a) shows the computed bed increase over initial bed (gray area) at the end of the simulation when all the sediment deposit on the bed. As expected, it can be observed that much of the sediment settle out in the reservoir and the downstream area near the lock gate; the sediment-driven current front disappear at around $x = 1.77 \text{ m}$ position because all the particles are settled out. Fig. 9(b) shows the vector of the velocity at $t = 21 \text{ s}$; the distribution of the velocity is as expected, and no oscillation is observed at the wetting and drying front. In Fig. 9(b), it can be seen that the rectangular area on the downstream side of the gate contains the highest concentration, the concentration decreases gradually in the radial part due to the propagation and spreading.

The left column of Fig. 10 shows the comparisons between the computed and measured sediment deposit as a function of radial distance for different initial concentrations. Very good agreement is observed, particularly in the radial part of the flume. The right column of Fig. 10 shows the comparisons between the predicted and measured front positions as a function of time for different initial concentrations. A general good agreement between the computed and measured results can be observed. It can be seen that the model can correctly capture the trend of the front wave propagation. Both computed and measured results show that the speed of the current front decreases as it propagates downstream due to the sediment deposition which reduces the driving force for the turbidity current; the farthest distance the current that can travel in the radial flume is reduced if the lower initial concentration is utilized. Moreover, in Fig. 10, the simulated results of the proposed model are compared with the results in Bradford and Katopodes (1999), it shows that our proposed model yields more accurate results than the model developed by Bradford and Katopodes (1999).

### Table 1

<table>
<thead>
<tr>
<th>Runs</th>
<th>$u_0$ (cm/s)</th>
<th>$c_0$</th>
<th>$d$ (mm)</th>
<th>$C_D$</th>
<th>$\omega$ (cm/s)</th>
<th>$p$</th>
<th>$\rho_s/\rho_w$</th>
<th>$\psi_p$</th>
<th>Time (min)</th>
</tr>
</thead>
<tbody>
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<td>GLASSA2</td>
<td>8.3</td>
<td>0.00339</td>
<td>0.03</td>
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<td>0.085</td>
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<td>2.5</td>
<td>0.15</td>
<td>30</td>
</tr>
<tr>
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<td>0.00394</td>
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<td>0.01</td>
<td>0.085</td>
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<td>2.5</td>
<td>0.15</td>
<td>33</td>
</tr>
<tr>
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<td>0.03</td>
<td>0.01</td>
<td>0.085</td>
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<td>30</td>
</tr>
<tr>
<td>GLASSA8</td>
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<td>0.03</td>
<td>0.01</td>
<td>0.085</td>
<td>0.5</td>
<td>2.5</td>
<td>0.15</td>
<td>28</td>
</tr>
</tbody>
</table>

Fig. 8. Plan view sketch of the laboratory tank.

Fig. 9. Run with initial concentration $c = 0.00966$: (a) Computed bed level increase when the test is done; (b) Simulated velocities and concentration of the turbidity current at $t = 21 \text{ s}$.
4.4. Submarine canyon-fan over irregular dry bed

This numerical experiment is used to test the performance of the proposed model when solving problems involving wetting and drying over irregular topography and submarine canyon-fan transition, i.e. internal hydraulic jump.

A submarine canyon followed by an initial dry plain is modeled over a 15 m \times 20 m domain, which consists of a 4 m long inclined bed with a slope of 0.08 in the x-direction, followed by a 11 m long plain. As shown in Fig. 11, three humps are defined on this submarine plain by:

\[
Z(x, y) = \max\{0 \text{ m}, 0.7 \text{ m} - 0.6 \cdot \{(x - 6 \text{ m})^2 + (y - 10 \text{ m})^2\},
0.2 \text{ m} - 0.1 \cdot \{(x - 8 \text{ m})^2 + (y - 5.5 \text{ m})^2\},
0.2 \text{ m} - 0.1 \cdot \{(x - 8 \text{ m})^2 + (y - 14.5 \text{ m})^2\}\}.
\]

The whole computational domain is initially dry and erodible. The inlet of turbidity flow is located in the center of the upstream sidewall as indicated in Fig. 11. The simulated parameters of the inflow are: \(h = 0.05 \text{ m}, u = v = 0 \text{ m/s}, c = 0.04\). Other parameters are: \(p = 0.5, d = 0.05 \text{ mm}, \rho_s = 2500 \text{ kg/m}^3, \rho_w = 1000 \text{ kg/m}^3, C_D = 0.004, \psi_p = 1, \tau_b = 1.6\).

Fig. 12 shows the evolution of the turbidity current over the initially dry plain with humps at different times. It shows that the hump close to the inclined bed with higher altitude is washed over at the beginning and it then splits the upstream turbidity current. Subsequently, other two humps with lower altitude are washed over and flooded. Wetting and drying processes over irregular to-
Fig. 12. Evolution of turbidity current over irregular dry bed (grey) at different times.

Fig. 13. Computed velocity vectors of turbidity current at different times.

pography are existing during the whole simulation, and no oscillations are generated at wet-dry fronts. This can be further verified by Fig. 13 which shows the vectors of current velocities at different times. No unphysical velocities are observed at the wetting and drying fronts during the whole simulation. This demonstrates the stability and robustness of the proposed numerical schemes.

Richardson number (Ri) is an important parameter governing the behavior of turbidity currents (Turner, 1979). A critical value 1 of Ri is generally used, so that the range Ri < 1 corresponds to a high-velocity supercritical turbid flow regime, and the range Ri > 1 corresponds to a low-velocity subcritical turbid flow regime. The change from supercritical flow to subcritical flow is accomplished via an internal hydraulic jump. Abrupt increase of water level near the inclined bed can be observed in Fig. 12 which indicates the internal hydraulic jump. In order to further investigate the transition of the simulated turbidity flow, the Richardson number is calculated at y = 9.5 m cross-section at different times, see Fig. 14 (a). It shows sharp increases of Ri from less than 0.2 to more than 2 at around x=5 m position. This demonstrates the ability of simulating the super- and sub-critical turbidity flow and capturing internal hydraulic jump by proposed numerical model. Fig. 14 (b) illustrates the bed changes at t = 540 s. It can be seen that most of the suspended sediment settle out on the sloped bed near the inlet, which is in accord with expectation.

5. Conclusion

A 2-D coupled numerical model is successfully developed to simulate sediment-driven turbidity current over irregular erodible bed with wetting and drying conditions. The proposed model is second order accurate in space and third order accurate in time. The model is built based on the complete mass and momentum conservation laws of the water-sediment mixture. After being verified by several numerical tests, the proposed model shows its robustness and satisfied accuracy. The developed numerical model incorporates two significant features: (i) One desired feature of the proposed models is well-balanced property which guarantees that the discretized bed-slope source terms exactly balance the numerical flux terms at steady status. In the developed numerical model, this property is successfully obtained by proposing a special discretization of bed-load source term. Furthermore, the special reconstruction of variables at wet-dry boundaries guarantee that the well-balanced property is also obtained at wetting and drying interfaces. The numerical tests clearly demonstrate the importance and necessity of the well-balanced property. (ii) Another desired feature of the proposed models is positivity preserving property which guarantees that the computed fluid depth and volumetric concentration of suspended sediment are non-negative in any time step during the simulation. In order to obtain this property, the
technique of tracking the wetting and drying front is first developed using non-negative bottom reconstruction and desingularization of point values. Based on the central-upwind method and developed wetting/drying technique, the positivity preserving property of the proposed numerical model is mathematically proved under a specific CFL condition which guarantees the positivity preserving property while maintaining the numerical stability.

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References


